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PARTS V & VI] SECTION A [VOL. 17

ON THE BENDING OF AN ELLIPTIC PLATE UNDER CERTAIN DISTRIBUTIONS OF LOAD. (II)*

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(Communicated by Prof. N. M. Basu, D.Sc.)

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In the present paper the author proposes to determine the deflexion of the central plane of a thin elliptic plate made of isotropic elastic material, clamped at the edge under the action of a load of weight W distributed uniformly over a half of the plate bounded by either of the principal axes.

Let the bounding ellipse of the central plane of the plate be given by,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (a>b>0)$$

If we introduce elliptic co-ordinates by the transformation

$$x+iy=c\cosh(\xi+i\eta). \qquad (c>0)$$

We have

$$x=c\cosh\xi\cos\eta$$
$$y=c\sinh\xi\sin\eta$$

Then ξ = const. and η = const. give a family of confocal ellipses and hyperbolas cutting each other at right angles. In particular $\xi=\alpha$ would represent the boundary, provided

$$a=c\cosh\alpha \text{ and } b=c\sinh\alpha.$$

*This forms a part of a thesis approved for the award of the Ph. D. degree of the University of Dacca. A few verbal and explanatory changes have been made here and there.

We shall consider two cases.

CASE I.

We shall first assume that the semi-ellipse

$$0 \leqslant \xi < \alpha$$
$$-\frac{\pi}{2} < \eta < \frac{\pi}{2}$$

is subjected uniformly to the load of weight W while the other half is free from pressure.

The pressure at any point of the semi-ellipse $0 \leqslant \xi < \alpha$; $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$ is then given by $p = \frac{2W}{\pi ab}$.

If we take the plate to be of thickness $2h'$ the deflexion w should satisfy the differential equation

$$DV_1^4 w = p$$

in the semi-ellipse $0 \leqslant \xi < \alpha$ and $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$ (where $D = \frac{2}{3} \cdot \frac{Eh'^3}{1-\sigma^2}$, E and σ being the Young's Modulus and Poisson's ratio respectively of the isotropic elastic material of which the plate is made.)
and $V_1^4 w = 0$ in the semi-ellipse.

$$0 \leqslant \xi < \alpha$$
$$\frac{\pi}{2} < \eta < \frac{3\pi}{2}.$$

Also it must satisfy the boundary conditions.

$$w = \frac{\partial \omega}{\partial \xi} = 0 \text{ over } \xi = \alpha.$$

Further the stresses must be continuous throughont the plate including the boundary.

Here we take the positive direction of the z-axis in the direction towards which the weight acts.

CASE II. Next we suppose that the semi-ellipse

$$0 < \xi < \alpha$$
$$0 < \eta < \pi$$

is subjected to the load.

If we denote the corresponding deflexion by $\bar{w}$ we must have

$$D\,\nabla_1^4\bar{w}=p$$

in the semi-ellipse

$$0<\xi<\alpha$$
$$0<\eta<\pi$$

and

$$\nabla_1^4\bar{w}=0$$

in the semi-ellipse $0<\xi<\alpha$; $\pi<\eta<2\pi$.

The stresses arising out of $\bar{w}$ must be continuous throughout the plate including the boundary.

Also $\bar{w}$ must satisfy the conditions

$$\bar{w}=\frac{\partial\bar{w}}{\partial\xi}=0 \quad \text{over } \xi=\alpha.$$

The problems in hand will be solved by combining the solutions of two known problems. Of these one is simple and its solution is well-known. The other is obtained by a slight modification of the solution of the problem of the bending of a thin semi elliptic plate under uniform pressure where the elliptic boundary is clamped and the straight boundary (which may either be the minor axis or the major axis) is freely supported. This latter problem was solved by Prof. Galerkin*.

The deflexion of the central plane of a thin elliptic plate with clamped edge under the action of a uniform pressure $\frac{1}{2}p$ at a point is given by

$$w_1=\frac{1}{16}\frac{p}{D}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\Big/\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)^{*}. \tag{1}$$

On the other hand Prof. Galerkin shows that the deflexion of the central plane of a semi-elliptic plate given by $0\leqslant\xi\leqslant\eta$; $-\pi/2\leqslant\eta\leqslant\pi/2$ whose elliptic boundary is clamped, and the straight boundary is freely

*Prof. B. G. Galerkin,—Messenger of Mathematics Vol. LII (1923) pp. 99-109.

**Elasticity by Love page 484 (Fourth edition)
The result in due of Prof. G. H. Bryan.

supported under uniform pressure $\frac{1}{2}p$ is given by

$$w_2 = L(3+4\cosh 2\xi+\cosh 4\xi)(3+4\cos 2\eta+\cos 4\eta) + (k_1\cosh\xi+A_1\cosh 3\xi)\cos\eta$$

$$+\sum_{n=2}^{\infty}\Big\{A_{2n-3}\cosh(2n-3)\xi+k_{2n-1}\cosh(2n-1)\xi + A_{2n-1}\cosh(2\eta+1)\xi\Big\}\cos(2n-1)\eta \quad . \quad . \quad . \quad (2)\dagger$$

where $L=\dfrac{pc^4}{3072\,D}$.

We can easily verify that the series for w_2 formally satisfies the differential equation

$$DV_1^4 w_2 = \tfrac{1}{2}p$$

at every point of the semi-ellipse $0\leqslant\xi<\alpha$, $-\frac{\pi}{2}<\eta<\frac{\pi}{2}$ excepting at the focus $\xi=0$, $\eta=0$.

The conditions for supported edge along the minor axis ($x=0$ or $\eta=\pm\pi/2$) are

$$w_2=0 \text{ and } G=0$$

where G is the expression for flexural couple arising out of w_2.

It is easy to show that the above conditions reduce to

$$w_2=0 \text{ and } \frac{\partial^2 w_2}{\partial\eta^2}=0$$

over $\eta=\pi/2$ and $\eta=-\pi/2$. And both these are formally satisfied by the series for w_2 over the minor axis.

The conditions for supported edge along the major axis may be shown to be

$$w=0 \text{ and } \frac{\partial^2 w}{\partial\eta^2}=0 \text{ over } \eta=0, \text{ and } \eta=\pi$$

and

$$w=0 \text{ and } \frac{\partial^2 w}{\partial\xi^2}=0 \text{ over } \xi=0 \ (0<\eta<\pi)$$

---

†Prof. Galerkin uses a slightly different but essentially the same form for the biharmonic part of the above expression.

Vide—B. G. Galerkin—Messenger of Mathematics—Vol. 52 (1923) p. 106.

Also remembering that*

$$f(\eta)=\frac{768}{\pi}\sum_{n=1}^{\infty}\frac{(-)^{n+1}\cos(2n-1)\eta}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}$$

where

$$f(\eta)=3+4\cos 2\eta+\cos 4\eta \quad \text{for } -\pi/2 \leqslant \eta \leqslant \pi/2$$

and $\quad f(\eta)=-(3+4\cos 2\eta+\cos 4\eta) \quad \text{for } \pi/2\leqslant \eta\leqslant 3\pi/2.$

The conditions $w_2=\frac{\partial w_2}{\partial \xi}=0$ over $\xi=\alpha$ are satisfied by the following values of k's and A's

$$k_1(\sinh 4\alpha+2\sinh 2\alpha)=\frac{K(\sinh 4\alpha-7\sinh 2\alpha)}{3.1.1.3.5} \tag{3}$$

$$A_1(\sinh 4\alpha+2\sinh 2\alpha)=-\frac{K\sinh 2\alpha}{1.1.3.5.} \tag{4}$$

$$K_{2n-1}(\sinh 4n\alpha+2n\sinh 2\alpha)$$
$$=-A_{2n-3}\{2\sinh(4n-2)\alpha+(2n-1)\sinh 4\alpha\}$$
$$-\frac{(-)^{n+1}K\{(2n-3)\sinh(2n+2)\alpha+(2n+5)\sinh 2n\alpha\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)} \tag{5}$$

$$A_{2n-1}(\sinh 4n\alpha+2n\sinh 2\alpha)$$
$$=A_{2n-3}\{\sinh(4n-4)\alpha+(2n-2)\sinh 2\alpha\}$$
$$+\frac{(-)^{n+1}K\{2n-5)\sinh 2n\alpha+(2n+3)\sinh(2n-2)\alpha\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)} \tag{6}$$

where $K=\frac{3072L\cosh^3\alpha}{\pi}=\frac{pc^4\cosh^3\alpha}{\pi D}$

Prof. Galerkin merely states that the equations obtained from the boundary conditions can be solved.

We may now assert that the function w_3 defined by

$$w_3=L(3+4\cosh 2\xi+\cosh 4\xi)(3+4\cos 2\eta+\cos 4\eta)$$
$$+(k_1\cosh\xi+A_1\cosh 3\eta)\cos\eta$$
$$+\sum_{n=2}^{\infty}\left\{A_{2n-3}\cosh(2n-3)\xi+k_{2n-1}\cosh(2n-1)\xi+A_{2n-1}\cosh(2n+1)\xi\right\}$$
$$\times\cos(2n-1)\eta \tag{7}$$

* Prof. B. G. Galerkin—The Messenger of Mathematics Vol. LII (1923) p. 107.

in the region $o \leqslant \xi \leqslant \alpha; -\frac{\pi}{2} \leqslant \eta \leqslant \frac{\pi}{2}$

and
$$w_3 = -L(3+4\cosh 2\xi + \cosh 4\xi)(3+4\cos 2\eta + \cos 4\eta) + (k_1 \cosh \xi + A_1 \cosh 3\xi)\cos\eta$$
$$+\sum_{n=3}^{\infty}\left\{A_{2n-1}\cosh(2n-3)\xi + k_{2n-1}\cosh(2n-1)\xi + A_{2n-1}\cosh(2n+1)\xi\right\} \times \cos(2n-1)\eta \qquad (8)$$

in the region $0 \leqslant \xi \leqslant \alpha$; $\frac{\pi}{2} \leqslant \eta \leqslant \frac{3\pi}{2}$

with the same values for L and k's and A's formally satisfies the equations

$$DV_1^4 w_3 = \tfrac{1}{2}p \quad \text{in } 0 \leqslant \xi \leqslant \alpha; -\frac{\pi}{2} > \iota > \frac{\pi}{2},$$

excepting at the focus $\xi = 0$, $\eta = 0$

and $D V_1^4 w_3 = -\frac{1}{2}p$ in $0 \leqslant \xi \leqslant \alpha ; \frac{\pi}{2} < \eta < \frac{3\pi}{2}$,

excepting at the focus $\xi = 0$, $\eta = \pi$.

Also the boundary conditions $w_3 = \frac{\partial w_3}{\partial \xi} = 0$

are formally satisfied over the elliptic boundary $\xi = \alpha$.

We now consider the function

$$w = w_1 + w_3 \qquad (9$$

The function w formally satisfies the differential equation

$DV_1^4 w = p$, at every point of the region

$0 \leqslant \xi < \alpha, -\pi/2 < \eta < \pi/2$

excepting at the focus $\xi = 0$, $\eta = 0$.

It formally satisfies the equation

$V_1^4 w = 0$ at every point of the region

$0 \leqslant \xi < \alpha$; $\pi/2 < \eta < 3\pi/2$

excepting at the focus $\xi = 0, \eta = \pi$

Also it formally satisfies the conditions

$$w=\frac{\partial w}{\partial_\nu}=0 \text{ over } \xi=\alpha$$

ν being the direction of the inward drawn normal to the edge line $\xi=\alpha$.

w is therefore expected to represent the deflexion of the central place of a thin elliptic plate clamped along the edge $\xi=\alpha$ carrying a load of weight w spread uniformly over the semi-ellipse

$$0\leqslant\xi<\alpha;\ -\pi/2<\eta<\pi/2.$$

We have now to consider convergence of the series giving the deflexion of the elliptic plate.

w_1 stands for

$$\tfrac{1}{16}\frac{p}{D}\left(1-\frac{x^2}{b^2}-\frac{y^2}{b^2}\right)^2\Big/\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)$$

which is necessarily finite and continuous throughout the elliptic plate. It possesses finite and continuous partial derivatives of all orders with respect to x and y and also mixed derivatives. In fact derivatives of order higher than 4 are all zero and $D_1^4\ w_1=\frac{p}{2}$. Again

$$L(3+4\cosh 2\xi+\cosh 4\xi)\times(3+4\cos 2\eta+\cos 4\eta)$$
$$=\frac{64Lx^4}{c^4} \qquad (10)$$

Denoting this part by f and the series

$$(k_1\cosh\xi+A_1\cosh 3\xi)\cos\eta$$

$$\sum_{n=2}^{\infty}\{A_{2n-3}\cosh(2n-3)\xi+k_{2n-1}\cosh(2n-1)\xi$$
$$+A_{2n-1}\cosh(2n+1)\xi\}\cos(2n-1)\eta$$

by F.

The deflexion may be written as

$$w=w_1+f+F \qquad (11)$$

in region I i.e., the part of the plate for which $x \geqslant 0$ and

$$w = w_1 - f + F$$

in region II in the part of the plate for which $x \leqslant 0$.
Now the function

$$\psi = f \quad \text{for } x > 0$$

and

$$\psi = -f \text{ for } x < 0$$

is evidently continuous at $x=0$.

Partial derivatives of ψ with respect to y are all zero. Partial derivatives upto the third order of ψ with respect to x are continuous throughout. While the partial derivative of ψ of the fourth order with respect is x has a discontinuity at $x=0$. In fact

$$\frac{\partial^4\psi}{\partial x^4} = \frac{24\times 64L}{c^4} \quad \text{for } x > 0$$

and

$$\frac{\partial^4\psi}{\partial x^4} = -\frac{24\times 64L}{c^4} \quad \text{for } x < 0$$

so

$$D\nabla_1^4\psi = \frac{p}{2} \quad \text{for } x > 0$$

and

$$D\nabla_1^4\psi = -\tfrac{1}{2}p \text{ for } x < 0.$$

It follows also that contributions to stress resultants and stress couples due to this part of the deflexion function are finite and continuous throughout the elliptic plate. Moreover they vanish over the minor axis.

Putting $f_{2n-1} = \sinh 4n\alpha + 2n \sinh 2\alpha$ we have from the recurrence formulae obtained in the previous section

$$A_1 f_1 = -K\frac{\sinh 2\alpha}{1.1.3.5}$$

$$A_3 f_3 = A_1 f_1 - K\left[\frac{\sinh 4\alpha}{1.3.5.7} - \frac{\sinh 2\alpha}{1.1.3.5}\right]$$

$$A_5 f_5 = A_3 f_3 + K\left[\frac{\sinh 6\alpha}{3.5.7.9} + \frac{\sinh 4\alpha}{1.3.5.7}\right]$$

$$A_7 f_7 = A_5 f_5 - K\left[\frac{\sinh 8\alpha}{5.7.9.11} + \frac{\sinh 6\alpha}{3.5.7.9}\right]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$A_{2n-1} f_{2n-1} = A_{2n-3} f_{2n-3} + (-)^{n+1} K\left[\frac{\sinh 2n\alpha}{(2n-3)(2n-1)(2n+1)(2n+3)} + \frac{\sinh 2(n-1)\alpha}{(2n-5)(2n-3)(2n-1)(2n+1)}\right]$$

Adding

$$A_{2n-1}f_{2n-1}=\frac{(-)^{n+1}\,K\sinh 2r\alpha}{(2n-3)(2n-1)(2n+1)(2n+3)}$$

giving a neat formula for the value of A_{2n-1}.

$$viz,\qquad A_{2n-1}=\frac{(-)^{n+1}\,K\sinh 2r\alpha}{(2n-3)(2n-1)(2n+1)(2n+3)f_{2n-1}}. \tag{13}$$

Now, since $\alpha>0$,

$$f_{2n-1}=\sinh 4n\alpha+2n\sinh 2\alpha>\sinh 4n\alpha>0.$$

So, we can show that

$$|A_{2n-1}|<\frac{K}{(2n-3)(2n-1)(2n+1)(2n+3)\,e^{2n\alpha}} \tag{14}$$

for $n\geqslant 2$.

Also since

$$k_{2n-1}f_{2n-1}=-A_{2n-3}\{2\sinh(4n-2)\alpha+(2n-1)\sinh 4\alpha\}$$
$$\frac{K(-)^{n-1}\{(2n-3)\sinh(2n+2)\alpha+(2n+5)\sinh 2n\alpha\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}.$$

We can without difficulty show that

$$|k_{2n-1}|\leqslant\frac{\overline{G}}{(2n-5)(2n-3)(2n-1)(2n+1)\,e^{2n\alpha}} \tag{15}$$

for sufficiently large n, where $\overline{G}$ is a positive constant independent of n.

Now, take the series

$$F_1=\sum_{n=1}^{\infty}k_{2n-1}\cosh(2n-1)\xi\cos(2n-1)\eta$$

we have

$$|\,k_{2n-1}\cosh(2n-1)\xi\cos(2n-1)\eta\,|$$
$$\leqslant|k_{2n-1}|\cosh(2n-1)|\xi|$$

$$\leqslant\frac{\overline{G}\quad e^{-\alpha}}{(2n-5)(2n-3)(2n-1)(2n+1)}e^{(2n-1)(|\xi|-\alpha)}$$

for sufficiently large n.

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The series therefore converges absolutely and uniformly for all η and all ξ such that $-\alpha \leqslant \xi \leqslant \alpha$

It therefore converges absolutely and uniformly in both the variables ξ and η, throughout the elliptic plate including the boundary $\xi=\alpha$. F_1, the sum of the series is therefore continuous throughout the plate including the boundary. Also it is easy to see that $\frac{\partial F_1}{\partial \xi}$ is obtained by term-by-term differentiation of the series for F_1, for the differentiated series is likewise absolutely and uniformly convergent throughout the elliptic plate inlcuding the boundary. $\frac{\partial F_1}{\partial \xi}$ is moreover continuous throughout the plate including the boundary.

Take

$$F_2=\sum_{n=2}^{\infty} A_{2n-3}\cosh (2n-3)\xi \cos (2n-1)\eta.$$

Now,

$$\begin{aligned}
& \left| A_{2n-3}\cosh (2n-3)\xi \cos (2n-1)\eta \right| \\
\leqslant \; & |A_{2n-3}| \cosh (2n-3)\xi \\
\leqslant \; & \frac{K\cosh(2n-3)\xi}{(2n-5)(2n-3)(2n-1)(2n+1)e^{2n\alpha}} \\
\leqslant \; & \frac{Ke^{(2n-3)|\xi|}}{(2n-5)(2n-3)(2n-1)(2n+1)e^{2n\alpha}} \\
= \; & \frac{Ke^{(2n-3)(|\xi|-\alpha)}}{e^{3\alpha}(2n-5)(2n-3)(2n-1)(2n+1)} \quad \text{for sufficiently large } n.
\end{aligned}$$

$\sum_{n=2}^{\infty} A_{2n-3}\cosh (2n-3)\xi \cos (2n-1)\eta$ is therefore absolutely and uniformly convergent for all η and all ξ such that $|\xi| \leqslant \alpha$. The sum F_2 therefore is continuous throughout the plate including the boundary. Also $\frac{\partial F_2}{\partial \xi}$ is obtained by-term-by term differentiation, for the differentiated series is absolutely and uniformly convergent throughout the plate including the boundary.

The same remark applies to the series

$$F_3=\sum_{n=1}^{\infty} A_{2n-1}\cosh(2n-1)\xi\cos(2n-1)\eta$$

The value of $F=F_1+F_2+F_3$ and also $\frac{\partial F}{\partial \xi}$ for $\xi=\alpha$ is therefore obtained by putting $\xi=\alpha$ in the series for F and the series for $\frac{\partial F}{\partial \xi}$ respectively. This justifies the method by which the values of the constants k's and A's were obtained. It is plain that derivatives of F of all orders in ξ and η can be had by formal term-by-term differentiation of the series for F at all points in the interior of the ellipse $\xi=\alpha$. Partial derivatives with respect to ξ and η upto the second order can be had by formal term-by-term differentiation at all points of the elliptic plate including the boundary.

Following the plan adopted in a previous note, we can easily show that

$$F_1=\sum_{n=1}^{\infty} k_{2n-1}\cosh(2n-1)\xi\cos(2n-1)\eta$$

is equal to the real part of the series

$$\sum_{n=1}^{\infty} k_{2n-1}P_{2n-1}(z)$$

where $P_{2n-1}(z)$ is some polynomial of degree $(2n-1)$.

The series $\sum_{n=1}^{\infty} k_{2n-1}\cosh(2n-1)\xi\cos(2n-1)\eta$ is uniformly convergent in any closed region contained within the ellipse $\xi=\alpha$, the region of uniform convergence extending upto the boundary of the ellipse.

The same is true of the series

$$\sum_{n=1}^{\infty} k_{2n-1}\sinh(2n-1)\xi\sin(2n-1)\eta.$$

So the series

$$\sum_{n=1}^{\infty} k_{2n-1} P_{2n-1}(z)$$

represents an analytic function throughout the interior of the ellipse. The derivatives of the function represented by $\sum_{n=1}^{\infty} k_{2n-1} P_{2n-1}(z)$ can be had by term-by-term differentiation throughout the interior of the ellipse. Its real part, viz., F_1 is plane harmonic throughout the interior of the ellipse, i. e., it satisfies the equation

$$V_1^2 F_1 = 0 \quad \text{throughout} \quad 0 \leqslant \xi < \alpha.$$

All partial derivatives of F_1 with respect to x and y and also mixed derivatives exist and are continuous throughout the interior of the plate, thus giving continuous contribution to stress resultants and couples in that region.

The series

$$F_2 + F_3 = \sum_{n=2}^{\infty} A_{2n-3} \cosh(2n-3)\xi \cos(2n-1)\eta$$

$$+ \sum_{n=1}^{\infty} A_{2n-1} \cosh(2n+1)\xi \cos(2n-1)\eta$$

which may be written as

$$\sum_{n=1}^{\infty} A_{2n-1} \{ \cosh(2n+1)\xi \cos(2n-1)\eta$$

$$+ \cosh(2n-1)\xi \cos(2n+1)\eta \}$$

throughout the elliptic plate including the boundary, is the real part of the function defined by the series,

$$(x - iy) \sum_{n=1}^{\infty} A_{2n-1} P_{2n}(z)$$

where $P_{2n}(z)$ is some polynomial of degree $2n$ in z.

We easily prove that the series $\sum_{n=1}^{\infty} A_{2n-1} \; P_{2n}(z)$ defines an analytic

function throughout the interior of the elliptic plate. Partial Derivatives of the function defined by the series are continuous throughout the interior of the plate and in this region F_2+F_3 satisfies the equation

$$\nabla_1^4 (F_2+F_3) = 0.$$

Contribution to stress resultants and stress couples arising out of the part (F_2+F_3) are therefore finite and continuous throughout the interior of the plate. We have now only to check if these are finite and continuous upto the boundary of the plate.

Now we proved elsewhere that for sufficiently large n

$$| k_{2n-1} | \leqslant \frac{\overline{G}}{(2n-5)(2n-3)(2n-1)(2n+1)e^{2n\alpha}}$$

and $$| A_{2n-1} | \leqslant \frac{K}{(2n-3)(2n-1)(2n+1)(2n+3)e^{2n\alpha}}$$

where K, $\overline{G}$ are positive constants independent of n.

Consider the series

$$\sum_{n=1}^{\infty} k_{2n-1} (2n-1)^2 \cosh (2n-1)\xi \cos (2n-1)\eta$$

i.e., the series that is obtained by differentiating the series for

$$F_1 = \Sigma\, k_{2n-1} \cosh (2n-1)\xi \cos (2n-1)\eta$$

term by term twice with respect to ξ.

The general term of the derived series is

$$(2n-1)^2 k_{2n-1} \cosh (2n-1)\xi \cos (2n-1)\eta$$

on the other hand

$$\begin{aligned}
(2n-1)^2 \,| k_{2n-1} \cosh (2n-1)\xi \cos (2n-1)\eta | &\leqslant (2n-1)^2 \,| k_{2n-1} | \cosh (2n-1)\xi \\
&\leqslant \frac{(2n-1)^2 \overline{G} \cosh (2n-1)\xi}{(2n-5)(2n-3)(2n-1)(2n+1)e^{2n\alpha}} \\
&\leqslant \frac{\overline{G} e^{(2n-1)|\xi|}}{(2n-5)(2n-3)e^{2n\alpha}} \\
&= \frac{\overline{G} e^{-\alpha}}{(2n-5)(2n-3)} e^{(2n-1)(|\xi|-\alpha)}
\end{aligned}$$

The range of uniform convergence of the series therefore extends upto the boundary $\xi=\alpha$. So the term-by-term differentiation of the series is possible for all ξ which lie in the range $0\leqslant\xi<\alpha$ and also the left hand differentiation with respect to ξ is possible for $\xi=\alpha$. Further the sum of the derived series represents the corresponding differential co-efficient of the sum of the original series and is a function that is continuous upto the boundary $\xi=\alpha$. The same statement is true for all partial derivatives upto the second order with respect to ξ and η.

Thus stress resultants and couples depending on derivatives upto the second order are finite and continuous throughout the plate including the boundary.

Now the normal shearing stress across an arc s, is defined by

$$\mathcal{N}=-D\frac{\partial}{\partial\nu}V_1^2w$$

where ν is the direction of the normal to the curve.

The contribution to $\mathcal{N}$ by F_1 is zero.

On the other hand

$$\begin{aligned}V_1^2(F_2+F_3)=32/c^2[&A_1\cosh\xi\cos\eta+2A_3(\cosh\xi\cos\eta+\cosh 3\xi\cos 3\eta)\\&+\ldots\ldots\\&+nA_{2n-1}\{\cosh\xi\cos\eta+\cosh 3\xi\cos 3\eta+\ldots\\&\qquad+\cosh(2n-1)\xi\cos(2n-1)\eta\}\\&+\ldots\ldots]\end{aligned}$$

The above result is valid throughout the plate including the boundary excepting the foci.

Now across any curve

$$\xi=\alpha',\quad\text{where}\quad 0<\alpha'<\alpha$$

$$\partial/\partial\nu\, V_1^2\,(F_2+F_3)$$

$$=h\frac{\partial}{\partial\xi}V_1^2\,(F_2+F_3).$$

By formal term by term differention we have

$$\partial/\partial\xi \; V_1^2 (F_2+F_3)$$

$$\begin{aligned}=32/c^2\,[A_1 \sinh\xi\cos\eta+2A_3(\sinh\xi\cos\eta+3\sinh 3\xi\cos 3\eta]\\ +3A_5(\sinh\xi\cos\eta+3\sinh 3\xi\cos 3\eta+5\sinh 5\xi\cos 5\eta)\\ +\ldots\ldots\ldots\ldots\ldots\ldots\\ +nA_{2n-1}\{\sinh\xi\cos\eta+3\sinh 3\xi\cos 3\eta\\ +\ldots+(2n-1)\sinh(2n-1)\xi\cos(2n-1)\eta\}\\ +\ldots\ldots\ldots\ldots\ldots\ldots].\end{aligned}$$

In what follows we restrict our attention to values of $\xi \geqslant 0$.

For throughout the plate ξ can never assume negative values

$$\begin{aligned}n\,|A_{2n-1}\{\sinh\xi\cos\eta+3\sinh 3\xi\cos 3\eta+\ldots\\ +(2n-1)\sinh(2n-1)\xi\cos(2n-1)\eta\}|\\ \leqslant n|A_{2n-1}|\{\sinh\xi+3\sinh 3\xi+5\sinh 5\xi\\ +\ldots+(2n-1)\sinh(2n-1)\xi\}\\ \leqslant \tfrac{1}{2}\,n|A_{2n-1}|\,(e^{\xi}+3e^{3\xi}+5e^{5\xi}+\ldots+(2n-1)\,e^{(2n-1)\xi}).\end{aligned}$$

On the other hand we can show that in the range $0<\beta\leqslant\xi\leqslant\alpha$

$$0<e^{\xi}+3e^{3\xi}+5e^{5\xi}+\ldots+(2n-1)\,e^{(2n-1\xi)}$$

$$<\frac{4(2n+1)\,e^{(2n+3)\alpha}}{\Delta^2} \quad \text{where}$$

$$\nabla=(e^{2\alpha}-1).$$

In the above ring-shaped region, we have therefore

$$\begin{aligned}n/A_{2n-1}\{\sinh\xi\cos\eta+3\sinh 3\xi\cos 3\eta+\ldots\\ +\ldots+(2n-1)\sinh(2n-1)\xi\cos(2n-1)\eta|\\ \leqslant\frac{4n(2n+1)\,|A_{2n-1}|\,e^{(2n+3)\alpha}}{(e^{2\alpha}-1)^2}\\ \leqslant\frac{2K\,e^{3\alpha}}{(e^{2\alpha}-1)^2}\cdot\frac{1}{(2n-3)(2n-1)}.\end{aligned}$$

So the series for $\frac{\partial}{\partial\xi}\,V_1^2(F_2+F_3)$ converges absolutely and uniformly in the closed ring-shaped region $0<\beta\leqslant\xi\leqslant\alpha$ including the boundaries.

The sum of the series therefore is the corrcct value for $\frac{\partial}{\partial \xi} V_1^2 (F_2+F_3)$ throughout the ring-shaped region in which the function is finite and continuous. Thus the normal shearing stress N across an arc $\xi=$ const. is certainly continuous in the ring-shaped region including the boundaries.* It was previously proved that N is continuous in the open domain $0 \leqslant \xi < a$. We have therefore proved that N is finite and continuous throughout the plate including the boundary.

We have thus proved that

$$w = w_1 + f + F \text{ in } 0 \leqslant \xi \leqslant a; -\pi/2 \leqslant \eta \leqslant \pi/2$$

and

$$\overline{w} = w_1 - f + F \text{ in } 0 \leqslant \xi \leqslant a; \quad \pi/2 \leqslant \eta \leqslant 3\pi/2$$

represent a function continuous throughout the plate including the boundary. It satisfies the differential equations

$$D \nabla^4 w = p \text{ in } 0 \leqslant \xi \leqslant a; -\pi/2 < \eta < \pi/2$$

and

$$D \nabla w = 0 \text{ in } 0 \leqslant \xi < a; \quad \pi/2 < \eta < 3\pi/2.$$

Further the stress-resultant N and the stress couples G and H as derived from w are finite and continuous throughout the plate including the boundary $\xi = a$.

Further $$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \xi = a.$$

It is only at the points that lie on the minor axis that the differential equation of equilibrium has not been shown to hold. It was so, because the function defining the deflexion is such that the partial derivative of the fourth order with respect to x has a discontinuity across that line.

We therefore take a rectangle bounded by the straight lines.

$$x = -1, \qquad x = 1$$

$$y = y_2, \qquad y = y_1$$

where

$$-b < y_2 < y_1 < b$$

and $1(>0)$ is small enough to ensure that the above rectangle lies

*Likewise, we can prove that the normal shearing stress N across an arc $\eta=$ const, is continuous in the same ring-shaped region including the boundaries.

completely within the ellipse. The value of the integral $\int N ds$ taken along the edges of the above rectangle can easily be shown to equal $-\frac{2Wl(y_1-y_2)}{\pi ab}$ which is just the weight of that part of the proposed load which happens to be contained within the rectangle.. It follows that there is no concentrated point load or line load anywhere on the minor axis.

w as defined above, therefore, represents the deflexion of the central plane of a thin elliptic plate clamped along the edge ($\xi=\alpha$) carrying a load of weight W spread uniformly over the semi-ellipse

$$0\leqslant\xi<\alpha\ ;-\frac{\pi}{2}<\eta<\frac{\pi}{2}.$$

CASE II

The deflexion of the central plane of a thin elastic plate of the shape of a semi-ellipse bounded by the major axis (say the semi-ellipse $0\leqslant\xi\leqslant\alpha$; $0\leqslant\eta\leqslant\pi$) under the action of a uniform pressure $\frac{1}{2}\,p$ where the elliptic boundary is clamped and the straight boundary is merely supported has also been given by Prof. Galerkin. He takes

$$w_4=L(3-4\cosh 2\xi+\cosh 4\xi)(3-4\cos 2\eta+\cos 4\eta)$$
$$+(l_1\sinh\xi+B_1\sinh 3\xi)\sin\eta$$
$$\sum_{2n=1}^{\infty}\left\{B_{2n-3}\sinh(2n-3)\xi+l_{2n-1}\sinh(2n-1)\xi+B_{2n-1}\sinh(2n-1)\xi\right\}$$
$$\times\sin(2n-1)\eta^{*} \qquad (16)$$

where $L=\frac{pc^4}{3072D}$ and the constants l's and B's are to be determined.

---

* The expression

$$E=B_1\sinh 3\xi\sin\eta+\sum_{n=1}^{\infty}\left\{B_{2n-3}\sinh(2n-3)\xi+B_{2n-1}\sinh(2n+1)\xi\right\}$$
$$\sin(2n-1)\eta$$

is a formal solution of the differentiation in equation $\nabla_1^4 E=0$, in elliptic coordinates. It was given in a slightly different but essentially the same form by Prof. Galerkin. (Vide—B. Galerkin—Messenger of Mathematics Vol. 52 (1923) pp 108-9.) See also Timpe—Mathematische Zeitscrift Vol. 17 (1923) P. (92.)

The function w_4 formally satisfies the differential equation

$$D\nabla_1^4 w_4 = \tfrac{1}{2}p$$

at every point of the semi-ellipse

$$0<\xi<a$$
$$0<\eta<\pi.$$

It also satisfies $w_4 = \dfrac{\partial^2 w_4}{\partial\eta^2} = 0$ over $\eta=0$ and $\eta=\pi$ and also the conditions

$$w_4 = \frac{\partial^2 w_4}{\partial\xi^2} = 0 \text{ over } \xi=0.$$

The condition of being freely supported along the major axis is thus formally satisfied.

On the other hand remembering that *

$$f(\eta) = \frac{768}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\eta}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}$$

where

$$f(\eta) = 3-4\cos 2\eta+\cos 4\eta \quad \text{for } 0\leqslant\eta\leqslant\pi$$

and

$$f(\eta) = -(3-4\cos 2\eta+\cos 4\eta) \text{ for } \pi\leqslant\eta\leqslant 2\pi.$$

The conditions $w_4 = \dfrac{\partial w_4}{\partial\xi} = 0$ over $\xi=a$ are satisfied by the following sets of values for the constants L's and B's.

$$L_1(\sinh 4a-2\sinh 2a) = \frac{K'(\sinh 4a+7\sinh 2a)}{3.1.1.3.5} \tag{17}$$

$$B_1(\sinh 4a-2\sinh 2a) = -\frac{K'\sinh 2a}{1.1.3.5} \tag{18}$$

$$l_{2n-1}(\sinh 4na-2n\sinh 2a) = -B_{2n-3}\{2\sinh(4n-2)a-(2n-1)\sinh 4a\} - \frac{K'\{(2n-3)\sinh(2n+2)a-(2n+5)\sinh 2na\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)} \tag{19}$$

$$B_{2n-1}(\sinh 4na-2n\sinh 2a) = B_{2n-3}\{\sinh(4n-4)a-(2n-2)\sinh 2a\} + \frac{K'\{(2n-5)\sinh 2na-(2n+3)\sinh(2n-2)a\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)} \tag{20}$$

*Prof. B. G. Galerkin—Messenger of Mathematics. Vol. 52 (1923) p. 109.

where $$K' = \frac{3072 L \sinh^3 \alpha}{\pi} = \frac{pc^4 \sinh^3 \alpha}{\pi D}.$$

We now define the functions $w_5(\xi, \eta)$ as

$$w_5(\xi, \eta) = L(3-4\cosh 2\xi + \cosh 4\xi)(3-4\cos 2\eta + \cos 4\eta) + (l_1 \sinh \xi + B_1 \sinh 3\xi)\sin\eta$$
$$+ \sum_{n=2}^{\infty} \left\{ B_{2n-3} \sinh(2n-3)\xi + l_{2n-1}\sinh(2n-1)\xi + B_{2n-1}\sinh(2n+1)\xi \right\} \times \sin(2n-1)\eta \quad (21)$$

in the region $0 \leqslant \xi \leqslant \alpha;\ 0 \leqslant \eta \leqslant \pi$

and

$$w_5(\xi, \eta) = -L(3-4\cosh 2\xi + \cosh 4\xi)(3-4\cos 2\eta + \cos 4\eta) + (l_1 \sinh \xi + B_1 \sinh 3\xi)\sin\eta$$
$$+ \sum_{n=2}^{\infty} \{ B_{2n-3}\sinh(2n-3)\xi \pm l_{2n-1}\sinh(2n-1)\xi + B_{2n-1}\sinh(2n+1)\xi \} \sin(2n-1)\eta \quad (22)$$

in the region $0 \leqslant \xi \leqslant \alpha;\ \pi \leqslant \eta \leqslant 2\pi$
with the same values for L, l's and B's.
The function w_5 formally satisfies the equation

$$D\nabla_1^4 w_5 = \tfrac{1}{2}p \text{ in } 0 < \xi < \alpha;\ 0 < \eta < \pi$$

and $$D\nabla_1^4 w_5 = -\tfrac{1}{2}p \text{ in } 0 < \xi < \alpha;\ \pi < \eta < 2\pi.$$

Also from the manner in which the constants l's and B's have been determined, we find that

$w_5 = \dfrac{\partial w_5}{\partial \xi} = 0$ are formally satisfied over $\xi = \alpha$

we now define the function $\bar{w}$ as

$$\bar{w} = w_1 + w_5. \quad (23)$$

Thus the function $\bar{w}$ formally satisfies the equation

$$D\nabla_1^4 \bar{w} = p \text{ over } 0 < \xi < \alpha;\ 0 < \eta < \pi$$

and $$\nabla_1^4 \bar{w} = 0 \text{ over } 0 < \xi < \alpha;\ \pi < \eta < 2\pi$$

also the conditions $w = \dfrac{\partial \bar{w}}{\partial \xi} = 0$ over $\xi = \alpha$.

So $\overline{w}$ is expected to represent the deflexion of the central plane of an elliptic plate clamped at the edge supporting a load of weight W spread uniformly over the semi-ellipse

$$0<\xi<\alpha\ ;\ 0<\eta<\pi.$$

It is now necessary to examine the expression w which formally satisfies the requirements of the problem.

It is easy to show that

$$L\,(3-4\cosh 2\xi+\cosh 4\xi)\,(3-4\cos 2\eta+\cos 4\eta) = \frac{64\,Ly^4}{C^4}\,. \qquad (24)$$

Denoting the above by f' and the series

$$(l_1 \sinh \xi+B_1 \sinh 3\xi) \sin \eta$$

$$+\sum_{n=2}^{\infty}\Big\{B_{2n-3} \sinh (2n-3)\xi+l_{2n-1} \sinh (2n-1)\xi$$

$$B_{2n-1} \sinh (2n+1)\xi\Big\} \sin (2n-1)\eta$$

by F' we may write the expression for deflexion as

$$\overline{w}=w_1+f'+F' \text{ in } 0\leqslant\xi\leqslant\alpha\ ;\ 0\leqslant\eta\leqslant\pi \qquad (25)$$

$$(i.e., \text{ for } y\geqslant 0)$$

$$\overline{w}=w_1-f'+F' \text{ in } 0\leqslant\xi\leqslant\alpha ; \pi\leqslant\eta\leqslant 2\pi \qquad (26)$$

$$(i.e., \text{ for } y\leqslant 0).$$

Now, the function

$$\psi'=f' \quad \text{for} \quad y\geqslant 0$$

and

$$\psi'=-f' \quad \text{for} \quad y\leqslant 0$$

is evidently continuous at $y=0$.

Partial derivatives of ψ' with respect to x are all zero. Again partial derivatives upto the third order of ψ' with respect to y are continuous throughout the plate while the partial derivatives of the fourth order with respect to y has a finite discontinuity at $y=0$
It is not even defined for $y=0$. In fact

$$\frac{\partial^4\psi'}{\partial y^4}=\frac{1536\,L}{c^4} \text{ for } y>0$$

and
$$\frac{\partial^4\psi'}{\partial y^4}=-\frac{1536L}{C^4}\text{ for } y<0.$$

So $\quad DV_1{}^4\psi'=\tfrac{1}{2}p$ for $y>0$ *i.e.* in $0<\xi<\alpha;\ 0<\eta<\pi$

and $\quad DV^4{}_1\psi'=-\tfrac{1}{2}p$ for $y<0$ *i.e.* in $0<\xi<\alpha;\pi<\eta<2\pi$.

It is plain that contributions to stress resultants and stress couples due to ψ' are finite and continuous throughout the plate including the boundary.

We next break up the series F' as before, into three parts
$$F'=F_1'+F_2'+F_3'$$

where
$$F_1'=\sum_{n=1}^{\infty} l_{2n-1}\sinh(2n-1)\xi\sin(2n-1)\eta$$

$$F_2'=\sum_{n=2}^{\infty} B_{2n-3}\sinh(2n-3)\xi\sin(2n-1)\eta$$

$$F_3'=\sum_{n=1}^{\infty} B_{2n-1}\sinh(2n+1)\xi\sin(2n-1)\eta$$

we write
$$f'_{n-1}=\sinh 4n\alpha-2n\sinh 2\alpha.$$

Then
$$B_1f_1'=-\frac{K'\sinh 2\alpha}{1\cdot 1\cdot 3\cdot 5}$$
$$B_3f_3'=B_1f_1'+\frac{K'\{\sinh 4\alpha+7\sinh 2\alpha\}}{1\cdot 1\cdot 3\cdot 5\cdot 7}$$
$$B_5f_5'=B_3f_3'+\frac{K'\{\sinh 6\alpha-9\sinh 4\alpha\}}{1\cdot 3\cdot 5\cdot 7\cdot 9}$$
$$B_7f_7'=B_5f_5'+\frac{K'\{3\sinh 8\alpha-11\sinh 6\alpha\}}{3\cdot 5\cdot 7\cdot 9\cdot 11}$$

... ...

$$B_{2n-1}f'_{2n-1}=B_{2n-3}f'_{2n-3}$$
$$+\frac{K'\{(2n-5)\sinh 2n\alpha-(2n+3)\sinh(2n-2)\alpha}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}.$$

Adding we have
$$B_{2n-1}f_2{}'_{n-1}=\frac{K'\sinh 2n\alpha}{(2n-3)(2n-1)(2n+1)(2n+3)}$$

So

$$B_{2n-1}=\frac{K'\sinh 2n\alpha}{(2n-3)(2n-1)(2n+1)(2n+3)f'_{2n-1}}.$$

We can now prove that for sufficiently large n

$$|B_{2n-1}|\leqslant\frac{2K'}{(2n-3)(2n-1)(2n+1)(2n+3)e^{2n\alpha}} \tag{27}$$

Also since

$$l_{2n-1}\,f'_{2n-1}=-B_{2n-3}\{2\sinh(4n-2)\alpha-(2n-1)\sinh 4\alpha\}$$
$$-\frac{K'\{(2n-3)\sinh(2n+2)\alpha-(2n+5)\sinh(2n-2)\alpha\}}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}.$$

We can now show that, for sufficiently large n

$$|l_{2n-1}|\leqslant\frac{G'}{(2n-5)(2n-3)(2n-1)(2n+1)\,e^{2n\alpha}} \tag{28}$$

where G' is a positive constant independent of n.

We can easily prove that the series

$$F_1'=\sum_{n=1}^{\infty} l_{2n-1}\sinh(2n-1)\,\xi\,\sin(2n-1)\eta$$

$$F_2'=\sum_{n=2}^{\infty} B_{2n-3}\sinh(2n-3)\,\xi\,\sin(2n-1)\eta$$

$$F_3'=\sum_{n=1}^{\infty} B_{2n-1}\sinh(2n+1)\,\xi\,\sin(2n-1)\eta$$

each converges absolutely and uniformly throughout the elliptic plate including the boundary $\xi=\alpha$. Each series therefore represents a continuous function throughout the plate including the boundary. The sum of the three series, i.e., the series for F', thus, converges absolutely and uniformly throughout the plate including the boundary. It may, therefore, be re-arranged in any way we like and the sum which remains unaltered for any re-arrangement of terms represents a function which is continuous throughout the plate including the boundary.

It is quite easy to show that $\frac{\partial F'}{\partial \xi}$ can be obtained by term-by-term differentiation of the series for F' with respect to ξ for all points

of the plate including the boundary. This justifies the process which was employed to determine the values of the constants. Further the partial derivatives of all orders of F' with respect to ξ and η and also mixed derivatives of all orders may be obtained by term-by-term differentation of the series for F' at all points within the interior of the ellipse and in that open domain they represent continuous functions. Derivatives upto the second order are however continuous upto the boundary.

We can easily show that the complex series

$$\sum_{n=1}^{\infty} l_{2n-1} (\cosh \overline{2n-1}\, \xi \cos \overline{2n-1}\, \eta + i \sinh \overline{2n-1}\, \xi \sin \overline{2n-1})\, \eta$$

transforms itself into the following series

$$\sum_{n=1}^{\infty} l_{2n-1} P_{2n-1}(z)$$

where $P_{2n-1}(z)$ is some polynomial of degree $(2n-1)$ in z.

The latter series being uniformly convergent in the interior as well as the boundary of the ellipse $\xi = a$ represents an analytic function in the interior of the elliptic plate.

The series $F_1' = \sum_{n=1}^{\infty} l_{2n-1} \sinh (2n-1)\, \xi \sin(2n-1)\eta$

being the imaginary part of a function analytic in the interior of the plate, possesses continuous partial derivatives of all orders with respect to x and y and also continuous mixed derivatives in the same domain. So the contributions to the stress-resultant N and stress couples G and H arising out of F_1' are continuous throughout the interior of the plate.

The function F_1' is harmonic throughout the interior of the elliptic plate, i.e.

$$V_1^2 F_1' = 0$$

and so $\quad V_1^4 F_1' = 0 \quad$ throughout $0 \leqslant \xi < a$

The series

$$F_2'+F_3'=\sum_{n=1}^{\infty} B_{2n-1}\Big\{\sinh(2n+1)\xi \sin(2n-1)\eta$$

$$+\sinh(2n-1)\xi \sin(2n+1)\eta\Big\}$$

is the imaginary part of the function

$$(x-iy)\sum_{n=1}^{\infty} B_{2n-1}P_{2n}(z)$$

where $\sum_{n=1}^{\infty} B_{2n-1}P_{2n}(z)$ is a function analytic in the interior of the elliptic plate. Writing

$$\sum_{n=1}^{\infty} B_{2n-1}P_{2n}(z)=v+iv'$$

we have

$$F_2'+F_3'=xv'-yv.$$

It follows that

$$V_1^4\,(F_2'+F_3')=0 \qquad \text{throughout the interior of the elliptic plate } \xi=\alpha.$$

The contributions to the stress-resultant N and stress couples G and H arising out of $F_2'+F_3'$ are also continuous throughout the interior of the plate. Having proved that G, H and N are continuous throughout the interior of the plate we have only to consider their behaviour on the boundary.

We can easily show that the series obtained by differentiating the series for F_1', F_2' and F_3' twice with respect to ξ and η are uniformly convergent throughout the plate including the boundary $\xi=\alpha$. This shows that the stress couples G and H, depending as they do on derivatives upto the second order are continuous upto the boundary. The region of continuity of the stress-resultant N may as in the previous section be also shown to extend upto the boundary of the plate $\xi=\alpha$.

It therefore follows that the series for $\bar{w}$ defines a function that is continuous throughout the elliptic plate including the boundary. The stresses as derived from $\bar{w}$ are continuous throughout the plate including the boundary. Further $\bar{w}$ satisfies the differential equation

$$D V_1^4 = w\frac{2W}{\pi ab} \text{ in } 0 \leqslant \xi < \alpha \; ; 0 < \eta < \pi$$

and the differential equation

$$V^4_1 \bar{w} = 0 \quad \text{in } 0 < \xi < \alpha; \; \pi < \eta < 2\pi.$$

Further let us take the rectangle

$$x = x_2; \; x = x_1$$
$$y = -l; \; y = l$$

where $-a < x_2 < x_1 < a$; and where l (>0) is small enough to ensure that the rectangle lies completely within the ellipse $\xi = \alpha$. The rectangle evidently contains the length $(x_1 - x_2)$ of the major axis. Now, the value of the integral $\int N \, ds$ taken along the edges of the rectangle may very easily be proved to be equal to $-\frac{2\,Wl\,(x_1 - x_2)}{\pi\, ab}$. This shows that there is no concentrated point load or a line load anywhere on the major axis.

Finally $w = \frac{\partial \bar{w}}{\sigma v} = 0$ are satisfied over the boundary $\xi = \alpha$.

Thus $\bar{w}$ represents the deflexion of the mid plane of a thin elliptic plate clamped along the edge $\xi = \alpha$, carrying a load of weight W spread uniformly over the semi-ellipse $0 < \xi < \alpha$, $0 < \eta < \pi$.

I am indebted to Prof. N. M. Basu for the kind interest he has taken in the preparation of the paper.

NOTE ON CIRCULAR CUBICS AND BICIRCULAR QUARTICS

By

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INTRODUCTION

The object of the present paper is to investigate, among other things, certain novel properties of a circular cubic and also of a bicircular quartic, with special reference to the normals, that can be drawn from the centres of inversion to the associated focal conics. Notice has also been taken of the rectangular hyperbola, that has been shown to pass through the feet of the afore-said normals.

The whole paper has been divided into three Sections, of which the first two deal respectively with a circular cubic and a bicircular quartic. Section III finally disposes of certain digressional matter on the construction of differential equations of the first order, whose singular solutions shall be *assigned* (plane) curves, e. g., a parabola or an ellipse or a hyperbola or a circular cubic or a bicircular quartic.

It has been tacitly assumed throughout the paper that the circular cubic or the bicircular quartic under discussion is of the *unrestricted* type, so that it is bicursal, i.e., has *unit* deficiency or *genus*. The *unicursal* type of circular cubic or bicircular quartic, being of a comparatively simple type, has been practically ignored in this paper.

Although the subject under discussion is classical in origin, the paper is believed to embody a considerable amount of original matter.

Other papers, bearing on this subject, are in course of preparation and are expected to be published in the near future.

SECTION I

(*Circular Cubics*)

Art. 1. As was pointed out by Dr. Casey, a bicursal circular cubic Γ possesses, in general, four circles of inversion $(\pi, \pi_1, \pi_2, \pi_3)$ and four focal parabolas $(\Sigma, \Sigma_1, \Sigma_2, \Sigma_3)$, so correlated that the cubic Γ is describable as the envelope of the set of circles, having their centres on any one of the focal conics (say Σ_r) and cutting the corresponding circle of inversion (say, π_r) at right angles. As a matter of fact, if any of the circles (say, π) and the associated parabola (Σ) be represented in the respective Cartesian forms:

$$(x-\alpha)^2+(y-\beta)^2=k^2 \quad \text{and} \quad y^2=4a(x+a) \quad . \quad . \quad . \quad (1)$$

the equation of the bicursal circular cubic can be thrown into the form:

$$x(x^2+y^2)+(2a-\alpha)(x^2+y^2)+(c^2-4a\alpha)x-4a\beta y$$
$$+\{2a(\alpha^2+\beta^2)-c^2\alpha\}=0, \qquad \text{(I)}$$

where

$$c^2\equiv k^2-\alpha^2-\beta^2. \quad . \quad . \quad . \quad (2)$$

By algebraic manipulations, (I) can be turned successively into each of the three equivalent forms, viz.

$$x(x^2+y^2)+(2a_r-\alpha_r)(x^2+y^2)+(c_r^2-4a_r\alpha_r)x-4a_r\beta_r y$$
$$+\{2a_r(\alpha_r^2+\beta_r^2)-c_r^2\alpha_r\}=0, \ (r=1, 2, 3), \quad . \quad . \quad . \quad \text{(II)}$$

provided that the constants $\{a_r\}$, $\{c_r\}$, $\{\alpha_r\}$, $\{\beta_r\}$ and $\{\lambda_r\}$ are determined by the relations:

$$\left.\begin{array}{l} a_r=a+\lambda_r, \quad \alpha_r=\alpha+2\lambda_r, \quad \beta_r=\dfrac{a\beta}{a+\lambda_r} \\ \text{and} \qquad c_r^2=c^2+4\lambda_r(2a+\alpha)+8\lambda_r^2, \end{array}\right\} \quad . \quad . \quad . \quad \text{(III)}$$

and λ_1, λ_2, λ_3 are the three *non-zero* roots of the biquadratic in λ, viz.:

$$\frac{a^2\beta^2}{a+\lambda}+(a-\lambda)(\alpha+2\lambda)^2-4a\lambda(\alpha+2\lambda)-c^2\lambda-a(\alpha^2+\beta^2)=0. \qquad \text{(IV)}$$

In view of (2) and (III), the equation for k_r is evidently

$$k_r^2=k^2-\alpha^2-\beta^2+(\alpha+2\lambda_r)^2+\frac{a^2\beta^2}{(a+\lambda_r)^2}$$
$$+4\lambda_r(2a+\alpha)+8\lambda_r^2, \ldots\ldots (r=1, 2, 3).$$

Comparison of the Cartesian equations (I) and (II) at once leads to the conclusion that the other three circles of inversion (π_1, π_2, π_3) and the corresponding focal parabolas $(\Sigma_1, \Sigma_2, \Sigma_3)$ are given by

$$(x-\alpha_r)^2+(y-\beta_r)^2=k_r^2 \quad \text{and} \quad y^2=4a_r(x+a_r), \ (r=1,2,3).$$

The very *forms* of the four focal parabolas $(\Sigma, \Sigma_1, \Sigma_2, \Sigma_3)$ at once proclaim their *confocal* character, the common focus, viz. O, being, of course, the double focus of the cubic Γ. The four centres of inversion, viz.

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \quad \text{and} \quad (\alpha, \beta)$$

will be henceforth denoted by the letters A, B, C, D (taken in order). The subsidiary relations, viz.

$$\left.\begin{array}{l} 2a_r-\alpha_r=2a-\alpha, \quad a_r\beta_r=a\beta, \quad c_r^2-4a_r\alpha_r=c^2-4a\alpha \\ 2a_r(\alpha_r^2+\beta_r^2)-c_r^2\alpha_r=2a(\alpha^2+\beta^2)-c^2\alpha, \quad \ldots\ldots. \end{array}\right\} (r=1,2,3) \quad \ldots \quad \text{(V)}$$

and

which follow automatically from (III), or as a consequence of (I) and (II) being *identical*, will be made use of in the succeeding articles.

Art. 2. Selecting indifferently one of the focal parabolas (say, Σ), viz.,

$$y^2=4a(x+a), \quad \ldots \quad \ldots \quad (1)$$

and writing

$$b=2a-\alpha,$$

we find, by Elementary Geometry, that the feet of the three normals drawn to Σ from the corresponding centre of inversion D (α, β), are the three *finite* * points of intersection of Σ with the rectangular hyperbola (Ξ), viz.,

$$xy+by-2a\beta=0, \quad \ldots \quad \ldots \quad (2)$$

whose centre (L) is the point $(-b, 0)$, and whose asymptotes, viz.,

$$x+b=0 \text{ and } y=0$$

are identical, respectively with the *real* asymptote of the cubic Γ and the *common* axis of $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$.

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* For brevity's sake, the expression *finite point* has been used to denote a point lying in the *finite* part of the plane, (as distinguished from a point lying on the line at infinity).

The *fourth* point of intersection of Σ and Ξ can be easily accounted for. For it is crystal-clear that, if K and K' be respectively the two points at infinity, attach-

In view of (V) of Art 1, the equation (2) of Ξ can at pleasure be re-written in each of three (equivalent) forms :

$$xy+(2a_r-\alpha_r)y-2a_r\beta_r=0,\ (r=1,2,3). \quad \dots \quad (3)$$

Interpreting this result geometrically, and noting that *more than four* arbitrary points cannot—unless specially related to one another—lie upon one and the same rectangular hyperbola, we may summarise our conclusions in the following manner :—

For a bicursal circular cubic Γ, the sixteen special points, consisting of the four centres of inversion A, B, C, D and of the feet of the four triads of normals drawn from them respectively to the four focal parabolas Σ_1, Σ_2, Σ_3, Σ must all lie on a certain rectangular hyperbola Ξ, whose asymptotes are respectively the real asymptote of Γ and the common axis of the four (confocal) focal parabolas.

Prior to the discussion of other geometrical features of the conic Ξ, we intend to make a short digression (in the next article) on the diverse types of polar conics of the original cubic Γ.

ing to the coordinate axes ($x=0$ and $y=0$), then (K, K') may as well be designated as the two points at infinity on Ξ. Remarkably, K is also the *real* point at infinity on Γ, (the other two points at infinity on Γ being the two circular points at infinity), whereas K' is the point of contact of Σ with the line at infinity. Thus this point K'—situated wholly at infinity—is the *fourth* point of intersection of Σ and Ξ, and is often ignored in the enumeration of points of intersection of the two conics. This explains the anomaly referred to.

According to (Euclidean) Projective Geometry, an *arbitrary* rt. line, lying in a plane, may *in a special sense* be said to be perpendicular to the line at infinity situated in that plane. The truth of this apparent paradox becomes manifest, if it is remembered that the condition of perpendicularity viz.

$$aa'+bb'=0$$

of two right lines, whose Cartesian equations are

$$ax+by+c=0 \quad \text{and} \quad a'x+b'y+c'=0$$

is fulfilled *independently* of a, b, if only $a'=0$, $b'=0$.

The afore-mentioned principle being borne in mind, it is clear that the line DK' is perpendicular to the line at infinity, which is, however, the tangent to Σ at K'. So DK' behaves like the normal to Σ at K'. This fourth normal (DK') from D to Σ being left out of consideration, we speak, as above, of only *three* (ordinary) normals (from D to Σ).

Art. 3. If $\phi(x,y)$ stands for the L.S of the equation (I) of Art. 1, the partial differential coefficients are given by

$$\left.\begin{aligned}&\frac{\delta\phi}{\delta x}=3x^2+y^2+2bx+(c^2-4a\alpha);\quad \frac{\delta\phi}{\delta y}=2(xy+by-2a\beta),\\ &\frac{\delta^2\phi}{\delta x^2}=2(3x+b);\quad \frac{\delta^2\phi}{\delta x\delta y}=2y;\quad \frac{\delta^2\phi}{\delta y^2}=2(x+b).\end{aligned}\right\}$$

We shall now invoke Art. 3 (Ch. VII, P. 93, Exs. 4, 6, 9) of Hilton's *Plane Algebraic Curves* (1919) to dispose of the sets of points, whose polar conics shall be of *prescribed* shapes.

Firstly, the locus (Λ) of points, whose polar conics are rectangular hyperbolas, is a right line, whose equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,\ i.e.,\ x=-\frac{b}{2}. \qquad \ldots \quad (1)$$

Thus Λ is parallel to the real asymptote ($x=-b$) of the given cubic Γ.

Secondly, the locus (S) of points, whose polar conics are parabolas, is the hyperbola, whose equation is

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\delta^2\phi}{\delta x\delta y}\right)^2=0,\ i.e.,\ (x+b)\ (3x+b)-y^2=0, \qquad \ldots \quad (2)$$

i.e.,

$$\frac{\left(x+\frac{2b}{3}\right)^2}{\left(\frac{b}{3}\right)^2}-\frac{y^2}{\left(\frac{b}{\sqrt{3}}\right)^2}=1.$$

So the centre of this hyperbola S is the point $M\left(-\frac{2b}{3},\ 0\right)$ and its eccentricity is 2.

Thirdly, the polar conic will be a circle, if the coordinates of the point in question conform to both the relations

$$\frac{\delta^2\phi}{\delta x^2}=\frac{\delta^2\phi}{\delta y^2}\ \text{and}\ \frac{\delta^2\phi}{\delta x\delta y}=0,$$

i.e., if the point coincides with the origin. That is to say, the double focus O (of Γ) represents the *only* point, whose polar conic (Γ') is a circle. This polar conic Γ'—of *circular* shape—will afterwards be referred to as the *polar circle* of the given circular cubic Γ. The actual equation to Γ' can be easily seen to be

$$b(x^2+y^2)+2\{(c^2-4a\alpha)x-4a\beta y\}+3\{2a(\alpha^2+\beta^2)-c^2\alpha\}=0, \quad \ldots \quad \ldots \quad (3)$$

so that its centre (E) is the point

$$\left(-\frac{c^2-4a\alpha}{b}, \frac{4a\beta}{b}\right), \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

provided, of course, that $b \neq 0$. When, however, $b=0$, the polar circle breaks up into the line at infinity and an ordinary line.

The symbol K being as before supposed to denote the *real* point at infinity on the given cubic Γ, and the symbol L representing the centre of the hyperbola Ξ, we shall now point out a sort of *reciprocal* relation between the points (O, E) and also between the points (K, L).

In the first place we remark that the polar conic of an arbitrary point (x', y') w.r.t. Γ has for its Cartesian equation:—

$$x'\{3x^2+y^2+2bx+(c^2-4a\alpha)\}+2y'(xy+by-2a\beta)$$
$$+[b(x^2+y^2)+2\{(c^2-4a\alpha)x-4a\beta y\}+3\{2a(\alpha^2+\beta^2)-c^2\alpha\}]=0. \quad \ldots \quad (5)$$

Accordingly the polar conic of the point E, as defined by (4), is given by

$$(c^2-4a\alpha)(3x^2+y^2)-b^2(x^2+y^2)-8a\beta xy=6ab(\alpha^2+\beta^2)$$
$$-3bc^2\alpha-(c^2-4a\alpha)^2-16a^2\beta^2,$$

which shews that the origin O is the centre.

Next, by (5), the polar conic of the point L $(-b, o)$ can be accommodated to the symbolic form:

$$Ax^2+2Gx+2Fy+C=0,$$

(where the co-efficients are all constants); manifestly this conic is a parabola, which has its axis parallel to the real asymptote $(x=-b)$ of the cubic Γ, and which may, in the phraseology of Projective Geometry, be spoken of as a *special* type of central conic, the special feature being that its centre is situated at infinity and is none other than the real point K of Γ at infinity.

To sum up the last set of results, we may simply state

(i) that the polar conic of any one of the two points (O, E) has the other for its centre;

and (ii) that the polar conic of any one of the two points (K, L) has the other for its centre.

Admittedly these results are to be regarded as simple illustrations of Casey's *general* principle that, whenever the polar conic of one point (w. r. t. a cubic) has another point for its centre, the polar conic of this latter point must have the former for its centre.

The accompanying diagram exhibits roughly the relative positions of the several points and lines (last mentioned), intimately related to a bicursal circular cubic Γ, it being implied that the double focus O of Γ does *not* lie on the real asymptote (so that $b=2a-\alpha\neq 0$.)

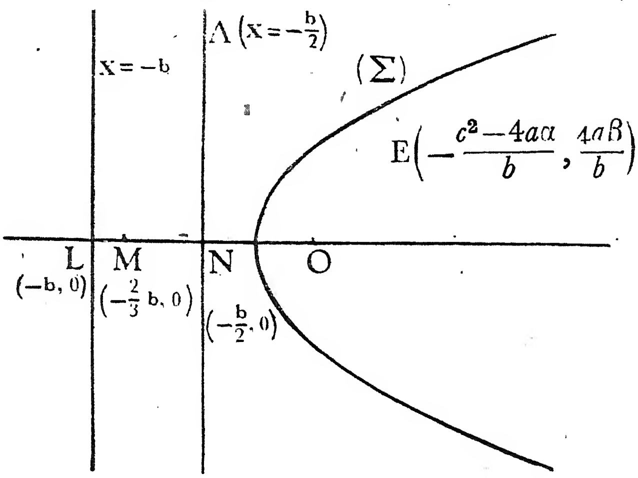

Fig. 1

In view of the results proved heretofore, the geometrical construction for the points L, M, N may be stated in the following *indirect* form:—

From the double focus O drop a perpendicular OL on the real asymptote of the given circular cubic Γ and bisect and trisect OL respectively at N and M, this latter point being the point of trisection nearer to the end-point L. Then the point L (i.e. the foot of the afore-said perpendicular) must be the centre of the rectangular hyperbola Ξ considered before. As for M, the characteristic property is that it is the centre of the hyperbola S, representing the locus of points, whose polar conics are parabolas. Finally, the point N has the distinctive property that the line, drawn through it parallel to the real asymptote of Γ, represents the locus of points, whose polar conics are rectangular hyperbolas.

The next article will be devoted to the consideration of the other type of circular cubic, whose double focus lies on its real asymptote, (so that $b=2a-\alpha=0$).

Art. 4. Suppose that Γ is a circular cubic, whose double focus O lies on its real asymptote. If I, J be the two circular points at infinity, and K denotes, as before, the real point (of Γ) at infinity, the tangents at I, J, K must be concurrent, the point of concurrence being the double focus O (as shewn in Fig. 2).

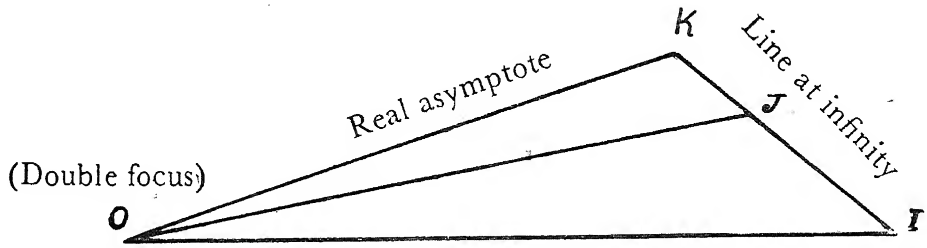

Fig. 2

I, J, K, being the points of contact of three of the tangents, that can be drawn from O to Γ, must then lie on the polar conic of O. Consequently the polar conic of O must consist of two right lines, one of which is the line at infinity IJK. The immediate inference is that the point O will lie on the Hessian (H) of Γ, and that the line at infinity will be a tangent to the Cayleyan.

Hence, observing that, when $b=0$, all the four points of Fig. 1, viz., O, L, M, N, coincide, we can give the following garb to the final result:—

A necessary condition for the double focus of a bicursal circular cubic Γ to lie on its real asymptote may be stated in any one of the following forms:—

(a) *that the double focus should lie on the Hessian;*

(b) *that the double focus should lie on the line-locus of points, whose polar conics are rectangular hyperbolas;*

(c) *that the double focus should be the centre of the rectangular hyperbola Ξ, which passes through the four centres of inversion as also*

through the feet of the four triads of normals, drawn respectively from the four points to the four related focal parabolas;

(*d*) *that the double focus should be the centre of the hyperbola* (S), *which represents the locus of points, whose polar conics are parabolas;*

(*e*) *that the polar circle,* (*i.e., the polar conic of the double focus*) *should break up into two right lines, one of which is the line at infinity;*

(*f*) *that the line at infinity should touch the Cayleyan of the cubic.*

It may not be out of place to remark that the class of bicursa circular cubics, having their double foci seated on their real asymptotes includes, within its fold, an important *sub-class*, consisting of *central* (and therefore bicursal) circular cubics. We observe incidentally that the converse of the property (a) is *not* necessarily true. In other words, when the double focus (of a bicursal circular cubic) lies on the Hessian, it may or may not lie on the real asymptote. The converse of each of the remaining properties (b), (c), (d), (e), (f) can be readily shown to be *valid*. That is to say, in order that the double focus of a bicursal circular cubic may lie on its real asymptote, the sets of conditions (b), (c), (d), (e), (f) are each necessary as well as sufficient, but the condition (a) is necessary, but by no means sufficient.

Art. 5. Let us now revert again to the *unrestricted* type of bicursal circular cubic Γ ($b\neq 0$). We now intend to establish another interesting property of the attached (rectangular) hyperbola Ξ.

It is common knowledge that, for a bicursal cubic Γ, the series of polar conics, answering to different points on a given right line W, constitute a *pencil*, passing through four *fixed* points, which are called the four *poles* of W. If we now look for the poles of the line at infinity *w. r. t.* the bicursal circular cubic Γ, given by (I) of Art. 1, we readily perceive that they are designable as the (four) points of intersection of the two conics:

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &\equiv 3x^2+y^2+2bx+(c^2-4a\alpha)=0 \\ \text{and}\quad \frac{\partial\phi}{\partial y} &\equiv 2(xy+by-2a\beta)=0 \end{aligned}\right\}, \text{ where } \quad b\equiv 2a-\alpha\neq 0.$$

Reference to Art. 2 makes it plain that the second of the two conics is no else than the rectangular hyperbola Ξ. Because the first conic is an ellipse, it follows that Ξ is the *only* (†) rectangular hyperbola that can be drawn though the four poles.

We are therefore entitled to summarise our conclusions as under:—

The rectangular hyperbola Ξ, that passes through the four centres of inversion of a bicursal circular cubic Γ and also through the feet of the four triads of normals drawn from the four centres to their respective focal parabolas, may also at pleasure be characterised

(i) *as the uniquely determinate rectangular hyperbola that can pass through the four poles of the line at infinity,*

or (ii) *as the uniquely determinate polar conic, having the real asymptote of Γ for one of its own asymptotes.*

SECTION II.

Bicircular Quartics

Art 6. A bicursal bicircular quartic Γ possesses—in precisely the same way as a (bicursal) circular cubic does—four circles of inversion Π, Π_1, Π_2, Π_3 and four corresponding focal conics Σ, Σ_1, Σ_2, Σ_3, so

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(‡) Attending to the patent facts that the number of equilateral hyperbolas, that can be drawn through four *given* points, is either 1 or ∞, and that any one of the four centres of inversion is the orthocentre of the triangle-formed by the other three, one can easily explain why the rectangular hyperbola Ξ is perfectly defined when it is said to pass through the four poles of the line at infinity, whereas the same curve (Ξ) is *imperfectly* defined—defined (as a matter of fact) as one among *infinitely many* rectangular hyperbolas—when it is said to pass through the four centres of inversion.

related that Γ may be described as the envelope of the set of circles, havingtheir centres located on any one of the focal conics (say, Σ_r) and intersecting the associated circle Π_r at right angles. The analogy between the two cases is almost complete save as to the fact that, whereas each of the focal conics of a circular cubic is *non-central* (i.e., parabolic), the focal conics of a bicircular quartic are all *central* (i.e., elliptic or hyperbolic). If any of the circles of inversion (say, Π) and its attached focal conic Σ be taken in the respective Cartesian forms:

$$(x-\alpha)^2+(y-\beta)^2=k^2 \qquad \text{and} \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \dots \quad \dots \quad \text{(I)}$$

the equation to Γ is easily obtained in the form:

$$(x^2+y^2+c^2)^2=4\{a^2(x-\alpha)^2+b^2(y-\beta)^2\}, \quad \dots \quad \dots \quad \text{(II)}$$

where $$c^2 \equiv k^2-\alpha^2-\beta^2.$$

By elementary algebraic manipulations, (II) can be converted into each of the *equivalent* forms, viz.,

$$(x^2+y^2+C_r^2)^2=4\{a_r^2(x-\alpha_r)^2+b_r^2(y-\beta_r)^2\}, \qquad (r=1, 2, 3) \qquad \text{(III)}$$

provided that the new triads of constants

$$\{a_r\}, \{b_r\}, \{c_r\}, \{\alpha_r\}, \{\beta_r\}$$

are defined by

$$c_r^2=c^2+\lambda_r, \; a_r^2=a^2+\frac{\lambda_r}{2}, \; b_r^2=b^2+\frac{\lambda_r}{2}, \; \alpha_r=\frac{2a^2\alpha}{2a^2+\lambda_r}, \; \beta_r=\frac{2b^2\beta}{2b^2+\lambda_r}, \qquad \text{(IV)}$$

and $\lambda_1, \lambda_2, \lambda_3$ are the three *non-zero* roots of the biquadratic in λ, viz.

$$\lambda^2+2c^2\lambda+4a^2\alpha^2+4b^2\beta^2-\frac{8a^4\alpha^2}{2a^2+\lambda}-\frac{8b^4\beta^2}{2b^2+\lambda}=0.$$

Comparison of the equations (II) and (III) at once points to the conclusion—which could be foreseen from other considerations—that the other three circles of inversion (Π_1, Π_2, Π_3) and the associated focal conics (Σ_1, Σ_2, Σ_2) are given by

$$(x-\alpha_r)^2+(y-\beta_r)^2=k_r^2 \text{ and } \frac{x^2}{a_r^2}+\frac{y^2}{b_r^2}=1, \qquad (r=1, 2, 3)$$

it being understood that k_r is defined by

$$c_r^2=k_r^2-\alpha_r^2-\beta_r^2,$$

i.e., $$k_r^2=c^2+\lambda_r+\frac{4a^4\alpha^2}{(2a^2+\lambda_r)^2}+\frac{4b^4\beta^2}{(2b^2+\lambda_r)^2}.$$

We note here, for future reference, the following subsidiary relations (which follow automatically from IV or from a comparison of II and III):—

$$a_r^2 - b_r^2 = a^2 - b^2,$$

$$a_r^2 \alpha_r = a^2 \alpha \quad \text{and} \quad b_r^2 \beta_r = b^2 \beta, \; (r = 1, 2, 3). \qquad \ldots \quad (\mathrm{V})$$

Denoting the four centres of inversion (α, β), (α_1, β_1), (α_2, β_2), (α_3, β_3) by the letters D, A, B, C, we shall investigate (in the next article) the tetrads of normals that can be drawn from the four points to the corresponding focal conics Σ, Σ_1, Σ_2, Σ_3.

Art. 7. If we now pick out at random one of the focal conics (say, Σ) having for its Cartesian equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we notice that the feet of the four normals, that can be drawn from the corresponding centre of inversion $D(\alpha, \beta)$ are the intersections of Σ, with the rectangular hyperbola Ξ, defined by

$$a^2 \alpha y - b^2 \beta x - (a^2 - b^2) xy = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

On the strength of (V) of Art. 6 the equation last written can be successively turned into the three *equivalent* forms:

$$a_r^2 \alpha_r y - b_r^2 \beta_r x - (a_r^2 - b_r^2) xy = 0. \qquad (r = 1, 2, 3). \qquad (2)$$

The obvious geometrical interpretation is that the sixteen points, viz., the feet of the four tetrads of normals, that can be drawn respectively from D, A, B, C, to the corresponding focal conics Σ, Σ_1, Σ_2, Σ_3 lie on one and the same rectangular hyperbola Ξ, which is completely definable by any one of the (equivalent) Cartesian equations (1), (2).

Regard being had to the undeniable fact that the hyperbola Ξ goes through the origin O, which is situated mid-way between the two real double foci $(\pm\sqrt{a^2 - b^2}, 0)$ of the quartic Γ, we can enunciate the final result in the undermentioned form :—

For a bicursal bicircular quartic Γ, the twenty special points, viz., the four centres of inversion, and the feet of the four tetrads of normals, drawn respectively from these centres to the four corresponding focal conics, lie on one

and the same rectangular hyperbola, whose asymptotes are respectively parallel and perpendicular to the line joining the two double foci of Γ, and which goes through the mid-point of this line.

SECTION III

(*Associated Differential Equations*)

Art. 8. A little reflection shows that, when a curve Γ of the most unrestricted type is given, there can, in general, be found an infinitude of *families* of curves, having Γ for their *common* envelope (complete or partial). Thus, for instance, the curve Γ is the envelope of:

(i)	its set of ∞^1 of	tangents,	(if $n \geqslant 2$),
(ii)	,, ,, ,,	circles of double contact,	(if $n \geqslant 2$)
(iii)	,, ,, ,,	osculating circles,	(,,)
(iv)	,, ,, ,, ,,	parabolas,	(,,)
(v)	,, ,, ,, ,,	rectangular hyperbolas,	(,,)
(vi)	,, ,, ,, ,,	conics,	(if $n \geqslant 3$)
(vii)	,, ,, ,, ,,	cubics	(if $n \geqslant 4$)
(viii)	,, ,, ,, ,,	quartics,	(if $n \geqslant 5$)

and so forth.

It is abundantly clear that, if the general Cartesian equation of any of the above families of curves be taken in the functional form :

$$f(x, y, c) = 0, \quad \ldots \quad \ldots \qquad (1)$$

the c – discriminant, say $F(x,y)$, when equated to zero and *freed from extraneous loci* (*if any*), will represent the original curve Γ. Furthermore, when the parameter c is eliminated from (1) by process of differentiation,

the resulting differential equation, viz: $\phi\ (x, y, p) = 0, \quad \left(p \equiv \frac{dy}{dx}\right) \ldots \quad (2)$

will have the same set of curves viz., (1) for its *integral curves*, and its p-discriminant, *freed* (*if need be*) *from extraneous loci*, will denote the original curve Γ. Thus *any given curve Γ may always be regarded as the*

common singular solution (complete or partial) of an infinite number of differential equations of the first order like (2).

In the succeeding article we shall illustrate the afore-mentioned principle with reference to a conic, a circular cubic and a bicircular quartic.

Art 9. There are a number of examples to be taken up one by one.

Example I. The parabola ($y^2=4ax$) is easily seen to be the *singular solution* of the differential equation:

$$(yp-2a)^2+y^2-4ax=0, \qquad (p\equiv\frac{dy}{dx}),$$

the general solution viz: $(x-t)^2+y^2-4ax=0$, (t is a parameter) representing the series of ∞^1 circles of double contact.

Example II. The ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

is the *common* singular solution of the two differential equations viz.,:

$$\left.\begin{aligned}&\left(\frac{py}{b^2}+\frac{x}{a^2}\right)^2-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0,\\ \text{and}\quad&\left(\frac{1}{p}\frac{x}{a^2}+\frac{y}{b^2}\right)^2-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0,\end{aligned}\right\}\quad\left(p\equiv\frac{dy}{dx}\right),$$

their respective *general* solutious, viz:

$$\left.\begin{aligned}&\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)(x-t)^2=0,\\ \text{and}\quad&\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(y-t)^2=0,\end{aligned}\right\}\quad(t \text{ is a parameter})$$

representing the two systems of circles of double contact that the conic can have. Evidently the corresponding result for a hyperbola is to be deduced by changing b^2 into $-b^2$.

Example III. For a bicursal circular cubic Γ, we may proceed as in Art. 1, and represent a circle of inversion π and its related focal parabola Σ in the respective forms:

$$(x-\alpha)^2+(y-\beta)^2=k^2 \qquad \text{and} \qquad y^2=4a\,(x+a)$$

and finally derive the equation of Γ in the symbolic form :

$$(x-\alpha)S+2a(y-\beta)^2=a, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

where
$$S\equiv x^2+y^2+2ax+c^2-2a\alpha$$
and
$$c^2\equiv k^2-\alpha^2-\beta^2.$$

Manifestly the family of circles, whose centres lie on Σ and which cut π orthogonally, has for its *functional equation:*

$$2at^2(x-a)-4at(y-\beta)-S=0, \quad \text{(where } t \text{ is a parameter)}. \quad \ldots \quad \text{(B)}$$

The *associated* differential equation is readily seen to be

$$\{S-2(x-\alpha)(x+py+a)\}^2-8a\{p(x-\alpha)-(y-\beta)\}$$
$$\{pS-2(y-\beta)(x+py+a)\}=0, \quad \ldots \quad \text{(C)}$$

where
$$p\equiv\frac{dy}{dx}.$$

Accordingly, this differential equation has (B) for its *general* solution, and (A) for its *singular* solution.

Repeating the same line of argument, we infer that the other three differential equations (of the first order), which claim, in common with (C), the original bicursal cubic Γ, viz., (A) for a *common* singular solution, are of the type :

$$\{S_r-2(x-\alpha_r)(x+py+a_r)\}^2-8a_r\{p(x-\alpha_r)-(y-\beta_r)\}$$
$$\{pS_r-2(y-\beta_r)(x+py+a_r)\}=0, \quad (r=1,2,3,),$$

it being understood that

$$a_r=a+\lambda_r, \; \alpha_r=\alpha+2\lambda_r, \; \beta_r=\frac{a\beta}{a+\lambda_r}, \quad c_r^2=c^2+4\lambda_r(2a+\alpha)+8\lambda_r^2,$$
$$S_r=x^2+y^2+2a_rx+c_r^2-2a_r\alpha_r,$$

and that $\lambda_1, \lambda_2, \lambda_3$ are the three *non-zero* roots of

$$\frac{a^2\beta^2}{a+\lambda}+(a-\lambda)(a+2\lambda)^2-4a\lambda(\alpha+2\lambda)-c^2\lambda-a(\alpha^2+\beta^2)=0.$$

It goes without saying that the corresponding *general solutions* are to be deduced from (B) by changing a, α, β, c, S respectively into $a_r, \alpha_r, \beta_r, c_r, S_r$.

Example IV. The equation of a bicursal bicircular quartic Γ being, as in Art. 5, taken in the form:

$$(x^2+y^2+c^2)^2=4\{a^2(x-\alpha)^2+b^2(y-\beta)^2\}, \; (c^2=k^2-\alpha^2-\beta^2), \quad \text{(I)}$$

where of course, a circle of inversion π and the related focal conic Σ are given by

$$(x-\alpha)^2+(y-\beta)^2=k^2 \qquad \text{and} \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

it is easy to verify that the *functional equation* of the series of circles of double contact, having their centres located on Σ, is

$$2a(x-\alpha)\cos t+2b(y-\beta)\sin t-(x^2+y^2+c^2)=0, \ (t \text{ is a parameter}) \qquad \ldots \quad \text{(II)}$$

whereas the corresponding *differential equation* is

$$a^2\{(x^2+y^2+c^2)-2(x-\alpha)(x+py)\}^2+b^2\{p(x^2+y^2+c^2)-2(y-\beta)(x+py)\}^2$$
$$-4a^2b^2\{p(x-\alpha)-(y-\beta)\}^2=0, \ \left(p\equiv\frac{dy}{dx}\right). \qquad \ldots \quad \ldots\text{(III)}$$

In other words, the bicircular quartic Γ, as given by (I), is the singular solution of (III), the general solution being, of course, (II).

Evidently the other three differential equations of the type (III), having (I) for their *common* singular solution, are given by

$$a_r^2\{(x^2+y^2+c_r^2)^2-2(x-\alpha_r)(x+py)\}^2+b_r^2\{p(x^2+y^2+c_r^2)-2(y-\beta_r)(x+py)\}^2$$
$$-4a_r^2b_r^2\{p(x-\alpha_r)-(y-\beta_r)\}^2=0, \ \left(p\equiv\frac{dy}{dx}\right), \qquad \text{(IV)}$$

it being postulated that $\{a_r\}$, $\{b_r\}$, $\{c_r\}$, $\{\alpha_r\}$ and $\{\beta_r\}$ are defined by

$$a_r^2=a^2+\frac{\lambda_r}{2}, \quad b_r^2=b^2+\frac{\lambda_r}{2}, \quad c_r^2=c^2+\lambda_r$$
$$\alpha_r=\frac{2a^2\alpha}{2a^2+\lambda_r}, \quad \beta_r=\frac{2b^2\beta}{2b^2+\lambda_r},$$

and $\lambda_1, \lambda_2, \lambda_3$ are the three *non-zero* roots of the biquadratic in λ, viz.,

$$\lambda^2+2c^2\lambda+4a^2\alpha^2+4b^2\beta^2-\frac{8a^4\alpha^2}{2a^2+\lambda}-\frac{8b^4\beta^2}{2b^2+\lambda}=0.$$

Needless to say, the *general* solutions of the three differential equations of the category (IV) are derivable from (II) by changing

$$a, \ b, \ c, \ \alpha, \ \beta$$

respectively into $\qquad a_r, \ b_r, \ c_r, \ \alpha_r, \ \beta_r.$

Thus, as is to be expected, the general solutions of the four differential equations (III) and (IV) are quite *distinct* from one another, although their *singular solution* is the same.

It is hardly necessary to mention that in each of the four examples I—IV, considered in this section, the singular solution represents only the envelope, there being no extraneous locus included in it.

MOTION IN FLUIDS OF VARIABLE DENSITY AND VARYING COEFFICIENT OF VISCOSITY

By

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This paper is in continuation with the auther's previous papers already published in the Proceedings of the National Academy of Sciences, India. In the previous papers motion in incompressible fluid either with varying coefficient of viscosity or variable density has been investigated. Here motion, when both density and viscosity vary at the same time, has been studied. We have treated here both of them as linearly varying and assumed $\rho=\rho_0+\lambda x$ and $\mu=\mu_0 \epsilon x$, where ρ_0 and μ_0 are the the values of ρ and μ respectively at the origin and λ and ϵ are constants. Here λ is a small quantity of the first order but ϵ not necessarily too small to be neglected.

The formula of stokes [1] for the resistance experienced by a slowly moving sphere in a fluid of constant density has been employed in physical researches of fundamental importance, as a means of estimating the size of minute globules of water and thence the number of globules contained in a cloud of given mass, and has led to much discussion both from the experimental and from the theoretical side. Oseen made an interesting innovation in the treatment of the above problem.

Here we have discussed the slow and steady motion, in the manner of Oseen, and solutions of hydrodynamical equations arising out of motion have been obtained in series.

Assuming the above laws of variation in ρ and μ and neglecting extraneous forces, the equations of motion are, in the method of Oseen,

$$
\left.\begin{aligned}
&(\rho_0+\lambda x)U\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}+(\mu_0+\varepsilon x)\ \nabla^2 u+\varepsilon\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right)-\frac{2}{3}\epsilon\left(\frac{u\partial}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\
&(\rho_0+\lambda x)U\frac{\partial v}{\partial x}=-\frac{\partial p}{\partial y}+(\mu_0+\varepsilon x)\ \nabla^2 v+\varepsilon\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\
&(\rho_0+\lambda x)U\frac{\partial w}{\partial x}=-\frac{\partial p}{\partial z}+(\mu_0+\varepsilon x)\ \nabla^2 w+\varepsilon\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\\
&\text{with the equation of continuity}\\
&\qquad\qquad\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{U\lambda}{\rho_0}
\end{aligned}\right\}(1)
$$

by writing $U+u$ for u and neglecting small quantities of the second order. These latter symbols now denote the components of the velocity which would remain if a translation $-U$ were superposed on the whole system. Putting $\bar{u}=u+\dfrac{U\lambda}{\rho_0}x$ in (1) we have

$$
\left.\begin{aligned}
&(\rho_0+\lambda x)U\frac{\partial \bar{u}}{\partial x}=-\frac{\partial p}{\partial x}+(\mu_0+\varepsilon x)\ \nabla^2\bar{u}+\varepsilon\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{u}}{\partial x}\right)+\\
&(\rho_0+\lambda x)U\frac{U\lambda}{\rho_0}-\frac{2\varepsilon U\lambda}{\rho_0}+\frac{2}{3}\frac{\epsilon U\lambda}{\rho_0}\\
&(\rho_0+\lambda x)U\frac{\partial v}{\partial x}=-\frac{\partial p}{\partial y}+(\mu_0+\varepsilon x)\ \nabla^2 v+\varepsilon\left(\frac{\partial v}{\partial x}+\frac{\partial \bar{u}}{\partial y}\right)\\
&(\rho_0+\lambda x)U\frac{\partial w}{\partial x}=-\frac{\partial p}{\partial z}+(\mu_0+\varepsilon x)\ \nabla^2 w+\varepsilon\left(\frac{\partial w}{\partial z}+\frac{\partial \bar{u}}{\partial z}\right)\\
&\text{with }\ \theta\equiv\frac{\partial \bar{u}}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.
\end{aligned}\right\}\ldots(1a)
$$

Let $u=\bar{u}_1+\bar{u}_2$, $v=v_1+v_2$ and $w=w_1+w_2$ such that

$$
\left.\begin{aligned}
&(\rho_0+\lambda x)U\frac{\partial \bar{u}_1}{\partial x}+\lambda U\bar{u}_1-\varepsilon\left(\frac{\partial \bar{u}_1}{\partial x}+\frac{\partial \bar{u}_1}{\partial x}\right)=-\frac{\partial p}{\partial x}+(\mu_0+\varepsilon x)\nabla^2\bar{u}_1\\
&(\rho_0+\lambda x)U\frac{\partial v_1}{\partial x}\qquad\qquad-\varepsilon\left(\frac{\partial v_1}{\partial x}+\frac{\partial \bar{u}_1}{\partial y}\right)=-\frac{\partial p}{\partial y}+(\mu_0+\varepsilon x)\nabla^2 v_1\\
&(\rho_0+\lambda x)U\frac{\partial w_1}{\partial x}\qquad\qquad-\varepsilon\left(\frac{\partial w_1}{\partial x}+\frac{\partial \bar{u}_1}{\partial z}\right)=-\frac{\partial p}{\partial z}+(\mu_0+\varepsilon x)\nabla^2 w_1\\
&\text{with }\ \theta_1\equiv\frac{\partial \bar{u}_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}=0.
\end{aligned}\right\}\ldots(2)
$$

and

$$(\rho_0+\lambda x)\,U\frac{\partial\bar{u}_2}{\partial x}-\lambda\,U\bar{\mu}_1-\varepsilon\left(\frac{\partial\bar{u}_2}{\partial x}+\frac{\partial\bar{u}_2}{\partial x}\right)=(\mu_0+\varepsilon x)\,\nabla^2\bar{u}_2+\frac{U\lambda}{\rho_0}\{(\rho_0+\lambda x)U-\tfrac{4}{3}\}$$

$$(\rho_0+\lambda x)\,U\,\frac{\partial v_2}{\partial x}\qquad-\varepsilon\left(\frac{\partial v_2}{\partial x}+\frac{\partial\bar{u}_2}{\partial y}\right)=(\mu_0+\varepsilon x)\,\nabla^2 v_2$$

$$(\rho_0+\lambda x)\,U\frac{\partial w_2}{\partial x}\qquad-\varepsilon\left(\frac{\partial w_2}{\partial x}+\frac{\partial\bar{u}_2}{\partial z}\right)=\mu_0+\varepsilon x)\,\nabla^2 w_2$$

$$\text{with}\qquad \theta_2\equiv\frac{\partial\bar{u}_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z}=0.$$

If there be a velocity potential $\bar{\phi}$ such that $\bar{u}_1=\dfrac{\partial\bar{\phi}}{\partial x}, v_1=-\dfrac{\partial\bar{\phi}}{\partial y}\ \&\ w_1=\dfrac{\partial\bar{\phi}}{\partial z}$ the equations in (2) are modified to

$$\left.\begin{aligned}
&(\rho_0+\lambda x)\,U\frac{\partial\bar{u}_1}{\partial x}+\lambda U\bar{u}_1-\varepsilon\left(\frac{\partial\bar{u}_1}{\partial x}+\frac{\partial\bar{u}_1}{\partial x}\right)=-\frac{\partial p}{\partial x}\\
&(\rho_0+\lambda x)\,U\frac{\partial v_1}{\partial x}\qquad-\varepsilon\left(\frac{\partial v_1}{\partial x}+\frac{\partial\bar{u}_1}{\partial y}\right)=-\frac{\partial p}{\partial y}\\
&(\rho_0+\lambda x)\,U\frac{\partial w_1}{\partial x}\qquad-\varepsilon\left(\frac{\partial w_1}{\partial z}+\frac{\partial\bar{u}_1}{\partial z}\right)=-\frac{\partial p}{\partial x}\\
&\text{with }\nabla^2\bar{\phi}=0
\end{aligned}\right\}\qquad\cdots\quad(4)$$

$$\text{provided}\qquad p=\{(\rho_0+\lambda x)\,U-2\varepsilon\}\frac{\partial\bar{\phi}}{\partial x}.\qquad\cdots\quad(5)$$

(3) may be put in the form

$$\left.\begin{aligned}
&(\mu_0+\varepsilon x)\,\nabla^2\bar{u}_2-\{(\rho_0+\lambda x)U-\varepsilon\}\frac{\partial\bar{u}_2}{\partial x}+\varepsilon\frac{\partial\bar{u}_2}{\partial x}=-\lambda U\bar{u}_1-\frac{U\lambda}{\rho_0}\{(\rho_0+\lambda x)U-\tfrac{4}{3}\epsilon\}\\
&(\mu_0+\varepsilon x)\,\nabla^2 v_2-\{(\rho_0+\lambda x)U-\varepsilon\}\frac{\partial v_2}{\partial x}+\varepsilon\frac{\partial\bar{u}_2}{\partial y}=0\\
&(\mu_0+\varepsilon x)\,\nabla^2 w_2-\{(\rho_0+\lambda x)U-\varepsilon\}\frac{\partial w_2}{\partial x}+\varepsilon\frac{\partial\bar{u}_2}{\partial z}=0\\
&\text{with }\theta_2=0.
\end{aligned}\right\}\cdots(3a)$$

Since the vortex lines must be circles having the axis of x as a common axis, we may assume

$$\xi=0,\ \eta=-\frac{\partial\chi}{\partial z}\text{ and }\zeta=\frac{\partial\chi}{\partial y}\qquad\cdot\quad\cdot\quad(6)$$

Using $\xi = \dfrac{\partial w_2}{\partial y} - \dfrac{\partial v_2}{\partial z}$, $\eta = \dfrac{\partial \bar{u}_2}{\partial z} - \dfrac{\partial w_2}{\partial x}$ and $\zeta = \dfrac{\partial v_2}{\partial x} - \dfrac{\partial \bar{u}_2}{\partial y}$ and after necessary modifications we get

$$\left.\begin{aligned} &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - \varepsilon\right\}\frac{\partial}{\partial x}\right]\eta = \varepsilon\nabla^2 w_2 - \lambda U\frac{\partial w_2}{\partial x} - \lambda U\frac{\partial \bar{u}_1}{\partial z} \\ &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - \varepsilon\right\}\frac{\partial}{\partial x}\right]\zeta = -\varepsilon\nabla^2 v_2 + \lambda U\frac{\partial v_2}{\partial x} + \lambda U\frac{\partial \bar{u}_1}{\partial y} \end{aligned}\right\} \quad (7)$$

Again, since $\theta_2 = 0$,

$$\left.\begin{aligned} &\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y} = \nabla^2 \bar{u}_2 \\ &\frac{\partial \zeta}{\partial x} - \frac{\partial \xi}{\partial z} = \nabla^2 v_2 \\ \text{and} \quad &\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x} = \nabla^2 w_2 \end{aligned}\right\}, \text{ we get (7)}$$

to be

$$\left.\begin{aligned} &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - 2\varepsilon\right\}\frac{\partial}{\partial x}\right]\eta = -\lambda U\frac{\partial w_2}{\partial x} - \lambda U\frac{\partial \bar{u}_1}{\partial z} \\ &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - 2\varepsilon\right\}\frac{\partial}{\partial x}\right]\zeta = \lambda U\frac{\partial v_2}{\partial x} + \lambda U\frac{\partial u_1}{\partial y} \end{aligned}\right\} \ldots (7a)$$

since $\xi = 0$.

$(7a)$ may again be simplified to the form

$$\left.\begin{aligned} &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - 2\varepsilon\right\}\frac{\partial}{\partial x} - \lambda U\right]\eta = -\lambda U\left(\frac{\partial \bar{u}_2}{\partial z} + \frac{\partial \bar{u}_1}{\partial y}\right) \\ &\left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - 2\varepsilon\right\}\frac{\partial}{\partial x} - \lambda U\right]\zeta = \lambda U\left(\frac{\partial \bar{u}_2}{\partial y} + \frac{\partial \bar{u}_1}{\partial y}\right) \end{aligned}\right\} \quad (7b)$$

From $7(b)$ we get

$$\begin{aligned} \left[(\mu_0+\varepsilon x)\nabla^2 - \left\{(\rho_0+\lambda x)U - 2\varepsilon\right\}\frac{\partial}{\partial x} - \lambda U\right]\chi &= \lambda U(\bar{u}_1 + \bar{u}_2) \\ &= \lambda U\bar{u} \quad \ldots\ldots (8) \end{aligned}$$

an additive function of x being obviously irrelevant.

Turning back to $(3a)$ we have

$$\left\{(\rho_0+\lambda x)U - \varepsilon\right\}\frac{\partial v_2}{\partial x} = (\mu_0+\varepsilon x)\nabla^2 v_2 + \varepsilon\frac{\partial \bar{u}_2}{\partial y}$$

$$= (\mu_0+\varepsilon x)\frac{\partial^2\chi}{\partial x\partial y}+\varepsilon\frac{\partial v_2}{\partial x}-\varepsilon\zeta.$$

$$\text{or}\left\{(\rho_0+\lambda x)U-2\varepsilon\right\}\frac{\partial v_2}{\partial x}=(\mu_0+\varepsilon x)\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial\chi}{\partial y}-2\varepsilon\frac{\partial\chi}{\partial y}$$

$$\text{or}\frac{\partial v_2}{\partial x}=\frac{1}{(\rho_0+\lambda x)U-2\varepsilon}\;\frac{\partial}{\partial x}\left\{(\mu_0+\varepsilon x)\frac{\partial x}{\partial y}\right\}-\frac{2\varepsilon}{(\rho_0+\lambda x)U-2\varepsilon}\frac{\partial x}{\partial y}$$

$$\left.\begin{aligned}\text{so that } v_2 &= \frac{\mu_0+\varepsilon x}{(\rho_0+\lambda x)U-2\varepsilon}\;\frac{\partial\chi}{\partial y}+\lambda U\int\frac{\mu_0+\varepsilon x}{\{(\rho_0+\lambda x)U-2\varepsilon\}^2}\frac{\partial\chi}{\partial y}dx\\ &\quad-2\varepsilon\int\frac{1}{(\rho_0+\lambda x)U-2\varepsilon}\frac{\partial\chi}{\partial y}dx\\ \text{And } w_2 &= \frac{\mu_0+\varepsilon x}{\rho_0+\lambda x)U-2\varepsilon}\;\frac{\partial\chi}{\partial z}+\lambda U\int\frac{\mu+\varepsilon\chi}{\{(\rho_0+\lambda x)U-2\varepsilon\}^2}\;\frac{\partial\chi}{\partial z}dx\\ &\quad-2\varepsilon\int\frac{1}{(\rho_0+\lambda x)U-2\varepsilon}\;\frac{\partial\chi}{\partial z}dx.\end{aligned}\right\}\ldots(9)$$

Now we have to determine χ to find u_2,v_2 and w_2. From (8) we get

$$\left[(\mu_0+\varepsilon x)\nabla^2-\left\{(\rho_0+\lambda x)U-2\varepsilon\right\}\frac{\partial}{\partial x}\right]\times$$

$$\left[(\mu_0+\varepsilon x)\nabla^2-\left\{(\rho_0+\lambda x)U-2\varepsilon\right\}\frac{\partial}{\partial x}-\lambda U\right]x=\lambda U\frac{\partial p}{\partial x},\text{ by (1}a)$$

$$=\lambda U\left[\{(\rho_0+\lambda x)U-2\varepsilon\}\frac{\partial^2\bar{\phi}}{\partial x^2}+\lambda U\frac{\partial\bar{\phi}}{\partial x}\right],\text{ by (5)}\quad.\;.\;.\;.\;\ldots(10)$$

Turning to the solution of the above equation, we observe that since the problem is one of preferential motion along the axis of x, we are justified in considering the solution of $\nabla^2\phi=0$ in the form $\phi=R.S$ where R is a function of x only and S that of ω and ϕ only where $\omega=\sqrt{y^2+z^2}$ and $\tan\phi=\frac{z}{y}$, so that from $\nabla^2\phi=0$ we get $\frac{1}{R}\frac{d^2R}{dx^2}+\frac{1}{S}\nabla_1{}^2S=0$ where $\nabla^2\equiv\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. This is satisfied if we write

$\frac{d^2R}{dx^2}-kR=0$ and $\nabla_1{}^2S+kS=0$ where k is an arbitrary constant. These give $R=c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}$, c_1 and c_2 being arbitrary constants, and

$$S^{(2)}=A_n J_n(\beta_1 w)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}n\phi,\text{ if }k=\beta_1{}^2$$

$$\text{or } B_m I_m (\beta_2 w) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \text{ if } k = -\beta_2^2.$$

In the differential equation (10) put $\chi = V.S.$ where V is a function of x only, so that

$$\left[(\mu_o+\varepsilon x)\nabla^2-\{(\rho_o+\lambda x)U-2\varepsilon\}\frac{\partial}{\partial x}\right]\left[(\mu_o+\varepsilon x)\left(\frac{d^2V}{dx^2}-kV\right)\right.$$
$$\left.-\{(\rho_o+\lambda x)U-2\varepsilon\}\frac{dV}{dx}-\lambda UV\right]S$$
$$=\lambda U\left[\{(\rho_o+\lambda x)U-2\varepsilon\}\frac{d^2R}{dx^2}+\lambda U\frac{dR}{dx}\right]S$$

Further simplification leads to

$$(\mu_o+\varepsilon x)^2\frac{d^4V}{dx^4}-2(\mu_o+\varepsilon x)\{(\rho_o+\lambda x)U-3\varepsilon\}\frac{d^3V}{dx^3}-$$
$$\left[2k(\mu_o+\varepsilon x)^2+3\lambda U(\mu_o+\varepsilon x)-\{(\rho_o+\lambda x)U\ 2\epsilon\}\{(\rho_o+\lambda x)U-3\varepsilon\}\right]\frac{d^2V}{dx^2}$$
$$+\left[2k(\mu_o+\varepsilon x)\{(\rho_o+\lambda x)U-3\varepsilon\}+2\lambda U\{(\rho_o+\lambda x)U-2\varepsilon\}\right]\frac{dV}{dx}$$
$$+\left[k^2(\mu_o+\varepsilon x)^2+k\lambda U(\mu_o+\varepsilon x)+\varepsilon k\{(\rho_o+\lambda x)U-2\varepsilon\}\right]V$$
$$=\lambda U\left[\lambda U\frac{dR}{dx}-k\{(\rho_o+\lambda x)U-2\varepsilon\}R\right] \quad . \quad . \quad . \quad (11)$$

Putting $\mu_o+\varepsilon x=\varepsilon x_1$, $\rho_o/\varepsilon=\rho_1$, $\lambda/\varepsilon=\lambda_1$, $\mu_o/\varepsilon=\mu_1$ and thus $\rho_1+\lambda_1 x=\rho_1-\lambda_1\mu_1+\lambda_1 x_1=\rho_2+\lambda_1 x_1$ (say), in (11) we get

$$x_1^2\frac{d^4V}{dx_1^4}-2x_1\{\lambda_1 Ux_1+(\rho_2 U-3)\}\frac{d^3V}{dx_1^3}$$
$$+\left[(\lambda_1^2U^2-2k)x_1^2+\lambda_1 U(2\rho_2 U-5-3)x_1+\rho_2^2U^2-5\rho_2 U+6\right]\frac{d^2V}{dx_1^2}$$
$$+2\left[k\lambda_1 Ux_1^2+(k\rho_2 U-3k+\lambda_1^2U^2)x_1+\lambda_1 U(\rho_2 U-2)\right]\frac{dV}{dx_1}$$
$$+k\left[kx_1^2+2\lambda_1 Ux_1+(\rho_2 U-2)\right]V=\lambda_1 U\left[\lambda_1 U\frac{dR}{dx_1}-k\{\lambda_1 Ux+(\rho_2 U-2)\}R\right]$$
$$=R^{x_1}\ (\text{say}) \quad . \quad . \quad . \quad (12)$$

First let us solve

$$x_1^2\frac{d^4V}{dx_1^4}-2x_1\left\{\lambda_1 Ux_1+(\rho_2U-3)\right\}\frac{d^3V}{dx_1^3}$$
$$+\left\{(\lambda_1^2U^2-2k)x_1^2+2\lambda_1U(\rho_2U-4)x_1+(\rho_2U-2)(\rho_2U-3)\right\}\frac{d^2V}{dx_1^2}$$
$$-2\left[k\lambda_1Ux_1^2+\{\lambda_1^2U^2+k(\rho_2U-3)\}x_1+\lambda_1U(\rho_2U-2)\right]\frac{dV}{dx_1}$$
$$+k\left\{kx_1^2+2\lambda_1Ux_1+(\rho_2U-2)\right\}V=0 \quad . \quad . \quad . \quad (13)$$

or $D_1V=0$, where D_1 Stands for the operator of V.

Construct an expression of the form

$$W=\sum_{n=0}^{\infty}c_n\,x_1^{\alpha+n}$$

$$D_1W=\Sigma\, c_0\,[\alpha(\alpha-1)\{(\alpha-2)(\alpha-3)-2(\alpha-2)(\rho_2U-3)$$
$$+(\rho_2U-2)(\rho_2U-3)\}x_1^{\alpha-2}$$
$$+\alpha x_1^{\alpha-1}\{-2\lambda_1U(\alpha-1)(\alpha-2)+2\lambda_1U(\rho_2U-4)(\alpha-1)$$
$$+2\lambda_1U(\rho_2U-2)\}$$
$$+x_1^{\alpha}\{\alpha(\alpha-1)(\lambda_1^2U^2-2k)+2\alpha(\lambda_1^2U^2+k.\overline{\rho_2U-3})$$
$$+k(\rho_2U-2)\}$$
$$+x_1^{\alpha+1}.\,2k\lambda_1U.\,(\alpha+1)+k^2x_1^{\alpha+2}].$$

Put $f_1=\lambda_1^2U^2-2k$, $f_2=2(\lambda_1^2U^2+k.\overline{\rho_2U-3})$ and $f_3=k(\rho_2U-2)$,

so that

$$D_1W=c_0[\alpha(\alpha-1)(\alpha-\rho_2U)(\alpha+1-\rho_2U)x_1^{\alpha-2}-2\lambda_1U\alpha^2(\alpha+1-\rho_2U)x_1^{\alpha-1}$$
$$+\{f_1\alpha(\alpha-1)+f_2\alpha+f_3\}x_1^{\alpha}+2k\lambda_1U(\alpha+1)x_1^{\alpha+1}+k^2x_1^{\alpha+2}]$$
$$+c_1[(\alpha+1)\alpha(\alpha+1-\rho_2U)(\alpha+2-\rho_2U)x_1^{\alpha-1}-2\lambda_1U(\alpha+1)^2(\alpha+$$
$$2-\rho_2U)x_1^{\alpha}+\{f_1(\alpha+1)\alpha+f_2(\alpha+1)+f_3\}x_1^{\alpha+1}$$
$$+2k\lambda_1U(\alpha+2)x_1^{\alpha+2}+k^2x_1^{\alpha+3}]$$
$$+c_2[(\alpha+2)(\alpha+1)(\alpha+2-\rho_2U)(\alpha+3-\rho_2U)x_1^{\alpha}-2\lambda_1U(\alpha+2)^2(\alpha+$$
$$3-\rho_2U)x_1^{\alpha+1}+\{f_1(\alpha+2)(\alpha+1)+f_2(\alpha+2)+f_3\}x_1^{\alpha+2}$$
$$+2k\lambda_1U(\alpha+3)x_1^{\alpha+3}+k^2x_1^{\alpha+4}]$$
$$+c_3[(\alpha+3)(\alpha+2)(\alpha+3-\rho_2U)(\alpha+4-\rho_2U)x_1^{\alpha+1}-2\lambda_1U(\alpha+3)^2(\alpha+$$
$$4-\rho_2U)x_1^{\alpha+2}+\{f_1(\alpha+3)(\alpha+2)+f_2(\alpha+3)+f_3\}x_1^{\alpha+3}$$
$$+2k\lambda_1U(\alpha+4)x_1^{\alpha+4}+k^2x_1^{\alpha+5}]$$

$$+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+$$

$$+c_{n-2}[(\alpha+n-2)(\alpha+n-3)(\alpha+n-2-\rho_2U)(\alpha+n-1-\rho_2U)x_1^{\alpha+n-4}-2\lambda_1U(\alpha+n-2)^2(\alpha+n-1-\rho_2U)x_1^{\alpha+n-3}+\{f_1(\alpha+n-2)(\alpha+n-3)+f_2(\alpha+n-2)+f_3\}x_1^{\alpha+n-2}+2k\lambda_1U(\alpha+n-1)x_1^{\alpha+n-1}+k^2x_1^{\alpha+n}]$$

$$+c_{n-1}[(\alpha+n-1)(\alpha+n-2)(\alpha+n-1-\rho_2U)(\alpha+n-\rho_2U)x_1^{\alpha+n-3}-2\lambda_1U(\alpha+n-1)^2(\alpha+n-\rho_2U)x_1^{\alpha+n-2}+\{f_1(\alpha+n-1)(\alpha+n-2)+f_2(\alpha+n-1)+f_3\}x_1^{\alpha+n-1}+2k\lambda_1U(\alpha+n)x_1^{\alpha+n}+k^2x_1^{\alpha+n+1}]$$

$$+c_n[(\alpha+n)(\alpha+n-1)(\alpha+n-\rho_2U)(\alpha+n+1-\rho_2U)x_1^{\alpha+n-2}-2\lambda_1U(\alpha+n)^2(\alpha+n+1-\rho_2U)x_1^{\alpha+n-1}+\{f_1(\alpha+n)(\alpha+n-1)+f_2(\alpha+n)+f_3\}x_1^{\alpha+n}+2k\lambda_1U(\alpha+n+1)x_1^{\alpha+n+1}+k^2x_1^{\alpha+n+2}]$$

$$+c_{n+1}[(\alpha+n+1)(\alpha+n)(\alpha+n+1-\rho_2U)(\alpha+n+2-\rho_2U)x_1^{\alpha+n-1}-2\lambda_1U(\alpha+n+1)^2(\alpha+n+2-\rho_2U)x_1^{\alpha+n}+\{f_1(\alpha+n+1)(\alpha+n)+f_2(\alpha+n+1)+f_3\}x_1^{\alpha+n+1}+2k\lambda_1U(\alpha+n+2)x_1^{\alpha+n+2}+k^2x_1^{\alpha+n+3}]$$

$$+c_{n+2}[(\alpha+n+2)(\alpha+n+1)(\alpha+n+2-\rho_2U)(\alpha+n+3-\rho_2U)x_1^{\alpha+n}-2\lambda_1U(\alpha+n+2)^2(\alpha+n+3-\rho_2U)x_1^{\alpha+n+1}+\{f_1(\alpha+n+2)(\alpha+n+1)+f_2(\alpha+n+2)+f_3\}x_1^{\alpha+n+2}+2k\lambda_1U(\alpha+n+3)x_1^{\alpha+n+3}+k^2x_1^{\alpha+n+4}]$$

$$+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+\quad\cdots\quad+$$

Thus $D_1W=c_0\alpha(\alpha-1)(\alpha-\rho_2U)(\alpha+1-\rho_2U)x_1^{\alpha-2}$ provided that

(*i*) $(\alpha+1)\alpha(\alpha+1-\rho_2U)(\alpha+2-\rho_2U)c_1-2\lambda_1U\alpha^2(\alpha+1-\rho_2U)c_0=0$

or $\quad(\alpha+1)(\alpha+2-\rho_2U)c_1-2\lambda_1U\alpha c_0=0, if\ \alpha+1\neq\rho_2u$

(*ii*) $(\alpha+2)(\alpha+1)(\alpha+2-\rho_2U)(\alpha+3-\rho_2U)c_2-2\lambda_1U(\alpha+1)^2(\alpha+2-\rho_2U)c_1+\{f_1\alpha(\alpha-1)+f_2\alpha+f_3\}c_0=0$

(*iii*) $(\alpha+3)(\alpha+2)(\alpha+3-\rho_2U)(\alpha+4-\rho_2U)c_3-2\lambda_1U(\alpha+2)^2(\alpha+3-\rho_2U)c_2+\{f_1(\alpha+1)\alpha+f_2(\alpha+1)+f_3\}c_1+2k\lambda_1U(\alpha+1)c_0=0$

(*iv*) $(\alpha+n+2)(\alpha+n+1)(\alpha+n+2-\rho_2U)(\alpha+n+3-\rho_2U)c_{n+2}-2\lambda_1U(\alpha+n+1)^2(\alpha+n+2-\rho_2U)c_{n+1}+\{f_1(\alpha+n)(\alpha+n-1)+f_2(\alpha+n)+f_3\}c_n+2k\lambda_1U(\alpha+n)c_{n-1}+k^2c_{n-2}=0,$ for $n\geqslant 2$.

Assuming that $\frac{c_{n+1}}{c_n}\to l$ as $n\to\infty$, we have $l=0$ from the above expression. This shows that the series thus obtained will be absolutely and uniformly convergent for all values of x_1.

Now our indical equation is $\alpha(\alpha-1)(\alpha-\rho_2 U)(\alpha+1-\rho_2 U)=0.$

which gives $\alpha=1, 0\ \rho_2 U \& \rho_2 U-1$.

Case (i). If $\rho_2 U$ is not an integer, the four solutions are

$$V_1=\Big[W\Big]_{\alpha=1},\quad V_2=\left[\frac{\partial W}{\partial\alpha}\right]_{\alpha=0},\quad V_3=\Big[W\Big]_{\alpha=\rho_2 U}$$

and
$$V_4=\left[\frac{\partial W}{\partial\alpha}\right]_{\alpha=\rho_2 U-1}$$

Case (ii). If $\rho_2 U=0, 1$, or any other integer, the solutions may be modified according to the method of Frobenius.

Now we have to find the Particular Integral of (12).

Here
$$\Delta=\begin{vmatrix} V_1''' & V_2''' & V_3''' & V_4''' \\ V_1'' & V_2'' & V_3'' & V_4'' \\ V_1' & V_2' & V_3' & V_4' \\ V_1 & V_2 & V_3 & V_4 \end{vmatrix}$$

$$2\int\frac{\lambda_1 U x_1+(\rho_2 U-3)}{x_1}dx_1$$

and its value is $ce=cx_1^2(\rho_2 U-3)\,e^{2\lambda_1}\,Ux_1, \ldots c$ being a constant.

Let Δ_r be the minor of V_r''' for $r=1, 2, 3 \& 4$. Then the Particular Integral[3] is given by

$$\sum_{r=1}^{4} V_r\int\frac{R_{x_1}\Delta_r}{x_1^2\Delta}\,dx_1$$

The expression within the Integral is of the form

$\int x_1^{l_1}\,e^{\gamma x_1}\,dx_1$ and this can be solved.

Thus we get the solution of (12) to be

$$V=\sum_{r=1}^{4} V_r\left\{\bar{c}_r+\int\frac{R_{x_1}\Delta_r}{x_1^2\Delta}\,dx_1\right\}$$

Hence a complete solution of the problem under consideration can be found.

My grateful thanks are due to Prof. A. C. Banerji under whom I have worked out this problem.

References.

(1) Lamb : *Hydrodynamics.* 6th. *Edn*, 1932, § 342.
(2) Loc. cit. § 191.
(3) Forsyth : *A treatise on Differential Equations*, 4th Edn. 1914. § 75.

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